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The algebraic theory of the fundamental germ

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Abstract. This paper introduces a notion of fundamental group appropriate for laminations.

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Introduction

Let $\mathcal{L}$ be a lamination: a space modeled on a "deck of cards" $\mathbb{R}^n \times \mathsf{T}$, where T is a topological space and overlap homeomorphisms take cards to cards continuously in the deck direction T. One thinks of $\mathcal{L}$ as a family of manifolds, the leaves, bound by a transversal topology prescribed locally by T. Using this picture, many constructions familiar to the theory of manifolds can be extended to laminations via the ansatz:

> Replace manifold object A by a family of manifold objects $\{A_L\}$ existing on the leaves of $\mathcal{L}$ and respecting the transverse topology.

For example, one defines a smooth structure to be a family of smooth structures on the leaves in which the card gluing homeomorphisms occurring in a pair of overlapping decks vary transversally in the smooth topology. Continuing in this way, constructions over $\mathbb{R}$, such as tensors, de Rham cohomology groups, *etc.* may be defined.

Identifying those constructions classically defined over $\mathbb{Z}$ is not as straightforward, especially if one wishes to follow tradition and define them geometrically.

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To see why this is true, consider the case of an exceptionally well-behaved lamination: an inverse limit $\widehat{M} = \lim_{\longleftarrow} M_\alpha$ of manifolds by covering maps. Such a system induces a direct limit of de Rham cohomology groups, and there is a canonical map from this limit into the tangential cohomology groups $H^*(\widehat{M};\ \mathbb{R})$ with dense image. In fact, here one may use the system to define – by completion of limits – tangential homology groups $H_*(\widehat{M};\ \mathbb{R})$ as well. If one endeavors to use this point of view to define the groups π_1, $H_*(\cdot\ \mathbb{Z})$, $H^*(\cdot\ \mathbb{Z})$, the result is failure since the systems they induce have trivial limits. The purpose of this paper is to introduce for certain classes of laminations $\mathcal{L}$ a construction $[\![\pi]\!]_1(\mathcal{L}, x)$ called the fundamental germ, a generalization of π_1 which represents an attempt to address this omission in the theory of laminations.

The intuition which guides the construction is that of the lamination as irrational manifold. Recall that for a pointed manifold (M, x), the deck group of the universal cover $(\widetilde{M}, \tilde{x}) \to (M, x)$ – which may be identified with $\pi_1(M, x)$ – reveals through its action how to make identifications within $(\widetilde{M}, \tilde{x})$ so as to recover (M, x) by quotient. Let us imagine that we have disturbed the process of identifying π_1 orbits, so that instead, points in an orbit merely approximate one another through some auxiliary transversal space T. The result is that $(\widetilde{M}, \tilde{x})$ does not produce a quotient manifold but rather coils upon itself, perhaps forming a leaf (L, x) of a lamination $\mathcal{L}$. The germ of the transversal T about x may be interpreted as the failed attempt of (L, x) to form an identification topology at x. The fundamental germ $[\![\pi]\!]_1(\mathcal{L}, x)$ is then a device which records algebraically the dynamics of (L, x) as it approaches x through the topology of T. See Figure 1.

One might define an element of $[\![\pi]\!]_1(\mathcal{L}, x)$ as a tail equivalence class of a sequence of approaches $\{x_\alpha\}$, where $L \ni x_\alpha \to x$ through T. In this paper, the laminations under consideration (see §2) have the property that there is a group G acting on L in such a way that every approach is asymptotic to one of the form $\{g_\alpha x\}$, for $g_\alpha \in G$. We then define $[\![\pi]\!]_1(\mathcal{L}, x)$ as the set of tail equivalence classes of sequences of the form $\{g_\alpha h_\alpha^{-1}\}$, where $g_\alpha x,\ h_\alpha x \to x$ in T. A groupoid structure on $[\![\pi]\!]_1(\mathcal{L}, x)$ is defined by component-wise multiplication of sequences, and $\pi_1(L, x)$ is contained in $[\![\pi]\!]_1(\mathcal{L}, x)$ as a subgroup. In practice, $[\![\pi]\!]_1(\mathcal{L}, x)$ has no additional structure; but for reasonably well-behaved laminations such as inverse limit solenoids and linear foliations, it is a group. And in certain instances when the fundamental germ is not a group – *e.g.* the Reeb foliation and the Sullivan solenoid – the groupoid structure is easily computed. See §§3–7 for definitions and examples.

When $\mathcal{L} = M$ is a manifold (a lamination with one leaf), $[\![\pi]\!]_1(M, x)$ is equal to ${}^*\pi_1(M, x)$, the nonstandard version of $\pi_1(M, x)$: the group of tail equivalence

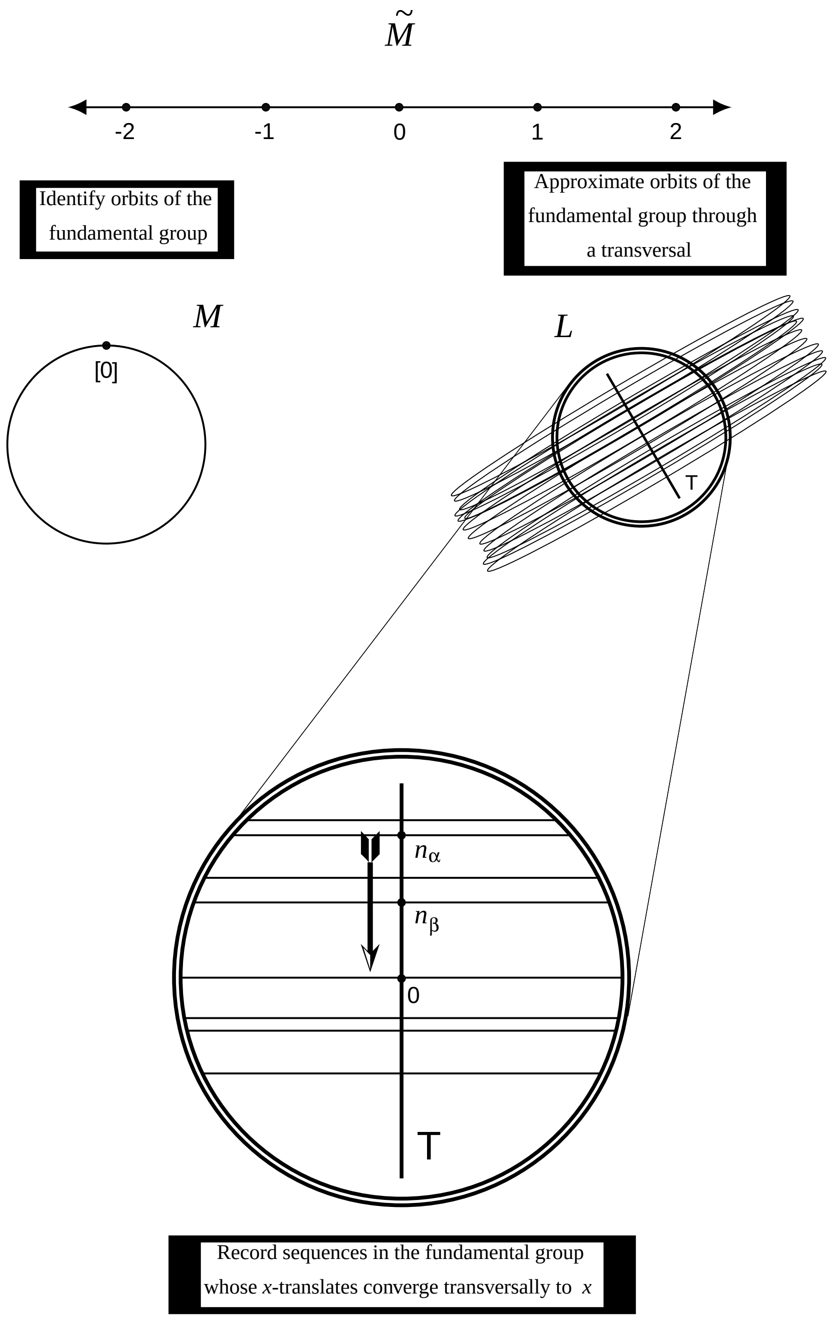

Figure 1: The Lamination as Irrational Manifold.

classes of *all* sequences in $\pi_1(M, x)$. When $\mathcal{L}$ is a lamination contained in a manifold M, under certain circumstances, §7, there is a map $[\![\pi]\!]_1(\mathcal{L}, x) \to {}^*\pi_1(M, x)$ whose image consists of those classes of sequences in $\pi_1(M, x)$ that correspond to the holonomy of $\mathcal{L}$. Thus, in expanding π_1 to its nonstandard counterpart, it is possible to detect – algebraically – sublaminations invisible to π_1.

One can profitably think of $[\![\pi]\!]_1(\mathcal{L}, x)$ as made from sequences of "G-diophantine approximations". In the case of an irrational foliation $\mathcal{F}_r$ of the torus $\mathbb{T}^2$ by lines of slope $r \in \mathbb{R} \setminus \mathbb{Q}$, §4.4, this is literally true: the elements of $[\![\pi]\!]_1(\mathcal{F}_r, x)$ are the equivalence classes of diophantine approximations of r. More generally, in $[\![\pi]\!]_1$ one finds an algebraic-topological tool which enables systematic translation of the geometry of laminations into the algebra of (non-linear) diophantine approximation.

One can extend the definition of the fundamental germ to include accumulations of L on points of other leaves. Thus if $\hat{x}$ is any point of $\mathcal{L}$, we define $[\![\pi]\!]_1(\mathcal{L}, x, \hat{x})$ as the set of classes of sequences of the form $\{g_\alpha \cdot h_\alpha^{-1}\}$ where $g_\alpha x, h_\alpha x \to \hat{x}$. We suspect that, together with the topological invariants of the leaves, the fundamental germs $[\![\pi]\!]_1(\mathcal{L}, x, \hat{x})$ will play an important role in the topological classification of laminations.

By unwrapping the accumulations of L implied by the fundamental germ $[\![\pi]\!]_1(\mathcal{L}, x)$, one obtains the germ universal cover $[\![\widetilde{\mathcal{L}}]\!]$, §9, which is a kind of nonstandard completion of $\widetilde{L}$. If $[\![\pi]\!]_1(\mathcal{L}, x)$ is a group, then under certain circumstances one may associate lamination coverings $\mathcal{L}_{\mathsf{C}} := \mathsf{C}\backslash[\![\widetilde{\mathcal{L}}]\!]$ of $\mathcal{L}$ to every conjugacy class of subgroup $\mathsf{C} < [\![\pi]\!]_1(\mathcal{L}, x)$, and when C is a normal subgroup, the quotient $[\![\pi]\!]_1(\mathcal{L}, x)/\mathsf{C}$ may be identified with the automorphism group of $\mathcal{L}_{\mathsf{C}} \to \mathcal{L}$. These considerations give rise to the beginnings of a Galois theory of laminations, §10.

This first paper on the fundamental germ is foundational in nature. One should not expect to find in it hard theorems, but rather the description of a complex and mysterious object which reveals the explicit connection between the geometry of laminations and the algebra of diophantine approximation. Due to its somewhat elaborate construction, we shall confine ourselves here to the following themes:

- Basic definitions: §§1–3.
- Examples: §§4–7.
- Functoriality: §8.
- Covering space theory: §§9,10.

The focus will be on laminations which arise through group actions: suspensions, quasi-suspensions, double coset foliations and locally-free Lie group actions. The exposition will be characterized by a careful exploration of a number of concrete examples which serve not only to illustrate the definitions in action but also to indicate the richness of the algebra they produce. In a second installment [5], to appear elsewhere, the construction of $[\![\pi]\!]_1$ will be extended to any lamination whose leaves admit a smooth structure.

1 Nonstandard Algebra

All ideas and statements in this section – with the exception of the notion of ultrascope – are classical and can be found in the literature. References: [8], [12].

Let $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathfrak{U} \subset 2^{\mathbb{N}}$ an ultrafilter all of whose elements have infinite cardinality. Given $S = \{S_i\}$ a sequence of sets and $X \in \mathfrak{U}$, write $S_X = \prod_{j \in X} S_j$. The *ultraproduct* is the direct limit

$$[S_i] \ := \ \varinjlim S_X,$$

where the system maps are the cartesian projections. If $S_i = S$ for all i, the ultraproduct is called the *ultrapower* of S, denoted ${}^*S$.

If S consists of nested sets, denote by ${}^{\circledcirc}S$ the set of sequences which converge with respect to S. For each $X \in \mathfrak{U}$, define a map $P_X : {}^{\circledcirc}S \to {}^{\circledcirc}S$ by restriction of indices: $P_X\big(\{x_\alpha\}\big) = \{x_\alpha\}|_{\alpha \in X}$. The *ultrascope* is the direct limit

$$\bigodot S_i \ := \ \varinjlim_{P_X} {}^{\circledcirc}S.$$

There is a canonical inclusion $[S_i] \hookrightarrow \bigodot S_i$, and when $S_i = S$ for all i, the ultrascope coincides with the ultrapower. In general, we have

$$\bigodot S_i = \bigcap {}^*S_i \supseteq {}^*\Big(\bigcap S_i\Big),$$

where the inclusion is an equality if and only if S_i is eventually equal to a fixed set.

If S is a (nested) sequence of groups or rings, the induced component-wise operations on sequences descend to operations making the ultraproduct (the ultrascope) a group or ring. This is also true if S is a (nested) sequence of fields: we remark here that the maximality property of ultrafilters is required to rule out zero divisors.

If one uses a different ultrafilter $\mathfrak{U}'$ and if S is a (nested) sequence of groups, rings or fields, then assuming the continuum hypothesis, it is classical [2] that the resulting ultraproduct is isomorphic to that formed from $\mathfrak{U}$. The same can shown for the ultrascope, however we shall not pursue this point here.

The ultrapower ${}^*\mathbb{R}$ is called *nonstandard* $\mathbb{R}$. There is a canonical embedding $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ given by the constant sequences, and we will not distinguish between $\mathbb{R}$ and its image in ${}^*\mathbb{R}$. For ${}^*x, {}^*y \in {}^*\mathbb{R}$, we write ${}^*x < {}^*y$ if there exists $X \in \mathfrak{U}$ and representative sequences $\{x_i\}$, $\{y_i\}$ such that $x_i < y_i$ for all $i \in X$. The non-negative nonstandard reals are defined ${}^*\mathbb{R}_+ = \{{}^*x \in {}^*\mathbb{R} \mid {}^*x \geq 0\}$. The Euclidean norm $|\cdot|$ on $\mathbb{R}$ extends to a ${}^*\mathbb{R}_+$-valued norm on ${}^*\mathbb{R}$. An element ${}^*x$ of ${}^*\mathbb{R}$ is called *infinite* if for all $r \in \mathbb{R}$, $|{}^*x| > r$, otherwise ${}^*x$ is called *finite*. ${}^*\mathbb{R}$ is a totally-ordered, non-archimedian field.

Here are two topologies that we may give ${}^*\mathbb{R}$:

- The *enlargement topology* ${}^*\tau$, generated by sets of the form ${}^*A$, where $A \subset \mathbb{R}$ is open. ${}^*\tau$ is 2^{nd}-countable but not Hausdorff.
- The *internal topology* $[\tau]$, generated by sets of the form $[A_i]$ where $A_i \subset \mathbb{R}$ is open for all i. $[\tau]$ is Hausdorff but not 2^{nd}-countable.

We have ${}^*\tau \subset [\tau]$, the inclusion being strict. It is not difficult to see that $[\tau]$ is just the order topology.

Proposition 1. $({}^*\mathbb{R},\ [\tau])$ *is a real, infinite dimensional topological vector space.*

We note however that $({}^*\mathbb{R}, +)$ is not a topological group with respect to ${}^*\tau$. Let ${}^*\mathbb{R}_{\text{fin}}$ be the set of finite elements of ${}^*\mathbb{R}$.

Proposition 2. ${}^*\mathbb{R}_{\text{fin}}$ *is a topological subring of* ${}^*\mathbb{R}$ *with respect to both the* ${}^*\tau$ *and* $[\tau]$ *topologies.*

The set of *infinitesimals* is defined ${}^*\mathbb{R}_\epsilon = \{{}^*\epsilon \mid |{}^*\epsilon| < M \text{ for all non-zero } M \in \mathbb{R}_+\}$, a vector subspace of ${}^*\mathbb{R}$. If ${}^*x - {}^*y \in {}^*\mathbb{R}_\epsilon$, we write ${}^*x \simeq {}^*y$ and say that ${}^*x$ is *infinitesimal* to ${}^*y$.

Proposition 3. ${}^*\mathbb{R}_{\text{fin}}$ *is a local ring with maximal ideal* ${}^*\mathbb{R}_\epsilon$ *and*

$$\frac{{}^*\mathbb{R}_{\text{fin}}}{{}^*\mathbb{R}_\epsilon} \cong \mathbb{R},$$

a homeomorphism with respect to the quotient ${}^*\tau$*-topology.*

We note that ${}^*\mathbb{R}_\epsilon$ is clopen in the $[\tau]$-topology; the quotient $[\tau]$-topology on ${}^*\mathbb{R}_{\rm fin}/{}^*\mathbb{R}_\epsilon$ is therefore discrete. ${}^*\mathbb{R}_\epsilon$ is not an ideal in ${}^*\mathbb{R}$. The vector space

$$ {}^\bullet\mathbb{R} \ := \ \frac{{}^*\mathbb{R}}{{}^*\mathbb{R}_\epsilon}, $$

is called the *extended reals*. By Proposition 3, ${}^\bullet\mathbb{R}$ contains a subfield isomorphic to $\mathbb{R}$.

Neither ${}^*\tau$ nor $[\tau]$ induce a satisfactory topology on ${}^\bullet\mathbb{R}$. Indeed, ${}^\bullet\mathbb{R}$ is not a topological vector space with respect to the topology induced by ${}^*\tau$, and the topology induced by $[\tau]$ makes $\mathbb{R} \subset {}^\bullet\mathbb{R}$ discrete. In §9 we will show that ${}^\bullet\mathbb{R}$ may be viewed as the universal cover of a host of 1-dimensional laminations, each one giving ${}^\bullet\mathbb{R}$ the structure of a topological vector space in which $\mathbb{R}$ has its usual topology.

Now let $\mathfrak{G}$ be any complete topological group. Some of the properties satisfied by ${}^*\mathbb{R}$ also hold for ${}^*\mathfrak{G}$. If τ denotes the topology of $\mathfrak{G}$, then the topologies ${}^*\tau$ and $[\tau]$ are defined exactly as above. ${}^*\mathfrak{G}$ is a topological group in the $[\tau]$ topology, but not in the ${}^*\tau$ topology. Denote by ${}^*\mathfrak{G}_\epsilon$ the classes of sequences converging to the unit element 1. ${}^*\mathfrak{G}_\epsilon$ is a group since a product of sequences converging to 1 in a topological group is again a sequence converging to 1. Let ${}^*\mathfrak{G}_{\rm fin}$ be the subgroup of ${}^*\mathfrak{G}$ all of whose elements are represented by sequences which converge to an element of $\mathfrak{G}$. We have the following analogue of Proposition 3:

Proposition 4. ${}^*\mathfrak{G}_\epsilon$ *is a normal subgroup of* ${}^*\mathfrak{G}_{\rm fin}$ *and*

$$ \frac{{}^*\mathfrak{G}_{\rm fin}}{{}^*\mathfrak{G}_\epsilon} \ \cong \ \mathfrak{G}, $$

a homeomorphism with respect to the quotient ${}^*\tau$*-topology.*

The left coset space

$$ {}^\bullet\mathfrak{G} \ := \ \frac{{}^*\mathfrak{G}}{{}^*\mathfrak{G}_\epsilon}, $$

is called the *extended* $\mathfrak{G}$. It contains $\mathfrak{G}$ as a subgroup. If $\mathfrak{G}$ is compact or abelian, then ${}^\bullet\mathfrak{G}$ is a group, though in general it need not be. We will avail ourselves of its natural structure as a ${}^*\mathfrak{G}$-set with respect to the left multiplication action.

2 Laminations associated to group actions

The laminations for which we shall define the fundamental germ arise from actions of groups: we review them here as a way of fixing notation. References: [1], [6], [7], [10].

Let us begin by reviewing the definitions and terminology surrounding the concept of a lamination. A *deck of cards* is a product $\mathbb{R}^n \times \mathsf{T}$, where T is a topological space. A *card* is a subset of the form $C = O \times \{\mathsf{t}\}$, where $O \subset \mathbb{R}^n$ is open and $\mathsf{t} \in \mathsf{T}$. A *lamination* of dimension n is a space $\mathcal{L}$ equipped with a maximal atlas $\mathcal{A} = \{\phi_\alpha\}$ consisting of charts with range in a fixed deck of cards $\mathbb{R}^n \times \mathsf{T}$, such that each transition homeomorphism $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ satisfies the following conditions:

(1) For every card $C \in \mathsf{Dom}(\phi_{\alpha\beta})$, $\phi_{\alpha\beta}(C)$ is a card.

(2) The family of homeomorphisms $\{\phi_{\alpha\beta}(\cdot, \mathsf{t})\}$ is continuous in t.

If T is totally disconnected, we say that $\mathcal{L}$ is a *solenoid*.

An open (closed) *transversal* in $\mathcal{L}$ is a subset of the form $\phi_\alpha^{-1}(\{x\} \times \mathsf{T}')$ where T' is open (closed) in T. Note that an open (closed) transversal need not be open (closed) in $\mathcal{L}$ *i.e.* if $\mathcal{L}$ is a manifold (viewed as a lamination with point transversals) then every point is an open transversal. An open (closed) *flow box* is a subset of the form $\phi_\alpha^{-1}(O \times \mathsf{T}')$, where O is open and $\mathsf{T}' \subset \mathsf{T}$ is open (closed). A *plaque* in $\mathcal{L}$ is a subset of the form $\phi_\alpha^{-1}(C)$ for C a card in the deck $\mathbb{R}^n \times \mathsf{T}$. A *leaf* $L \subset \mathcal{L}$ is a maximal continuation of overlapping plaques in $\mathcal{L}$. Note that $\mathcal{L}$ is the disjoint union of its leaves; we denote by L_x the leaf containing the point x. A lamination is *weakly minimal* if it has a dense leaf; it is *minimal* if all of its leaves are dense. A transversal which meets every leaf is called *complete*. Unless we say otherwise, all transversals in this paper will be complete and open. Two laminations $\mathcal{L}$ and $\mathcal{L}'$ are said to be *homeomorphic* if there is a homeomorphism $f : \mathcal{L} \to \mathcal{L}'$ mapping leaves homeomorphically onto leaves and transversals homeomorphically onto transversals.

2.1 Suspensions

Let B be a manifold, F a topological space and $\rho : \pi_1 B \to \mathsf{Homeo}(F)$ a representation. The *suspension* of ρ is the space

$$\mathcal{L}_\rho = \widetilde{B} \times_\rho F$$

defined by quotienting $\widetilde{B} \times F$ by the diagonal action of $\pi_1 B$, $\alpha \cdot (\tilde{x}, t) = (\alpha \cdot \tilde{x}, \rho_\alpha(t))$. The suspension is a fiber bundle over B with model fiber F. Conversely, if $E \to B$ is a fiber bundle with model fiber F a compact manifold, then any foliation of E transverse to the fibers is a suspension.

If $F = \mathfrak{G}$ is a topological group and $\varphi : \pi_1 B \to \mathfrak{G}$ a homomorphism, then the representation $\rho : \pi_1 B \to \mathsf{Homeo}(\mathfrak{G})$ defined $\rho_\gamma(g) = g \cdot \varphi(\gamma^{-1})$ gives rise to what we call a $\mathfrak{G}$-*suspension*, denoted $\mathcal{L}_\varphi$, a principle $\mathfrak{G}$-bundle over B.

The action of $\pi_1 B$ used to define $\mathcal{L}_\rho$ is properly discontinuous and leaf preserving, hence $\mathcal{L}_\rho$ is a lamination modeled on the deck of cards $\widetilde{B} \times F$. If $K = \ker(\rho)$ and L is a leaf, we have $K \trianglelefteq \pi_1 L$. $\mathcal{L}_\rho$ is minimal (weakly-minimal) if and only if every (at least one) $\rho(\pi_1 B)$ orbit is dense.

The restriction $p|_L$ of the projection $p : \mathcal{L}_\rho \to B$ to a leaf L is a covering map. Suppose that p_L is a Galois covering (we say that L is *Galois*). The deck group D_L of $p|_L$ has the property that

$$D_L \cdot x \;=\; L \cap F_x,$$

where F_x is the fiber of p through x. In particular, if we give $(L \cap F_x) \subset F_x$ the subspace topology, we have an inclusion

$$D_L \hookrightarrow \mathsf{Homeo}(L \cap F_x).$$

A manifold B is a suspension with F a point and $\rho : \pi_1 B \to F$ trivial. The following subsections discuss examples which are more interesting.

2.1.1 Inverse limit solenoids

Let $C = \left\{\rho_\alpha : M_\alpha \to M\right\}$ be an inverse system of pointed manifolds and finite Galois covering maps with initial object M; denote by

$$\widehat{M} \;=\; \widehat{M}_C \;:=\; \varprojlim M_\alpha$$

the limit. By definition $\widehat{M} \subset \prod M_\alpha$, so elements of $\widehat{M}$ are denoted $\hat{x} = (x_\alpha)$, where $x_\alpha \in M_\alpha$. The natural projection onto the base surface is denoted $p\colon \widehat{M} \to M$. We may identify the universal covers $\widetilde{M}_\alpha$ with $\widetilde{M}$ and choose the universal covering maps $\widetilde{M} \to M_\alpha$ to be compatible with the system C. By universality, there exists a canonical map $i : \widetilde{M} \longrightarrow \widehat{M}$.

Let $H_\alpha = (\rho_\alpha)_*(\pi_1 M_\alpha) < \pi_1 M$. Associated to C is the inverse limit of deck groups

$$\hat{\pi}_1 M \;:=\; \varprojlim \pi_1 M/H_\alpha,$$

a Cantor group since the $\pi_1 M/H_\alpha$ are finite. By universality of inverse limits, the projections $\pi_1 M \to \pi_1 M/H_\alpha$ yield a canonical homomorphism $\iota : \pi_1 M \to \hat{\pi}_1 M$ with dense image. The closures of the images $\iota(H_\alpha)$ are clopen, and give a neighborhood basis about 1. Let $\mathcal{L}_\iota$ be the associated $\hat{\pi}_1 M$-suspension.

Proposition 5. *$\widehat{M}$ is homeomorphic to $\mathcal{L}_\iota$. In particular, $\widehat{M}$ is a solenoid.*

Proof. Let $\Upsilon\colon \widetilde{M} \times \hat{\pi}_1 M \to \widehat{M}$ be the map defined $(\tilde{x}, \hat{g}) \mapsto \hat{g} \cdot i(\tilde{x})$. Υ is invariant with respect to the diagonal action of $\pi_1 M$, and descends to a homeomorphism $\widetilde{M} \times_\rho \hat{\pi}_1 M \to \widehat{M}$. □

2.1.2 Linear foliations of torii

Let V be a p-dimensional subspace of $\mathbb{R}^n$. Denote by $\widetilde{\mathcal{F}}_V$ the foliation of $\mathbb{R}^n$ by cosets $\mathbf{v}+V$. The image $\mathcal{F}_V$ of $\widetilde{\mathcal{F}}_V$ in the torus $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$ gives a foliation of the latter by euclidean manifolds. Since $\mathcal{F}$ is transverse to the fibers of some fibration $\mathbb{T}^n \to \mathbb{T}^p$, it is itself a suspension. This suspension structure may be made explicit as follows. Let $q=n-p$, and display V as the graph of a $q\times p$ matrix map,

$$\mathbf{R}:\mathbb{R}^p\to\mathbb{R}^q,$$

whose columns are independent. For $\mathbf{y}\in\mathbb{R}^q$, denote by $\bar{\mathbf{y}}$ its image in $\mathbb{T}^q$. Let $\varphi_{\mathbf{R}}:\mathbb{Z}^p\to\mathbb{T}^q$ be the homomorphism defined

$$\varphi_{\mathbf{R}}(\mathbf{n}) \;=\; \overline{\mathbf{Rn}},$$

and denote by $\mathcal{L}_{\varphi_{\mathbf{R}}}$ the corresponding $\mathbb{T}^q$-suspension. Then $\mathcal{F}_V\approx\mathcal{L}_{\varphi_{\mathbf{R}}}$. We note that the closure of any leaf of $\mathcal{F}_V$ is isomorphic to the closure of the image of V in $\mathcal{F}_V$, which is a torus of dimension m with $p\le m\le n$. In particular, $\mathcal{F}_V$ consists of noncompact leaves if and only if $m>p$.

2.1.3 Anosov foliations

Let $\Sigma=\mathbb{H}^2/\Gamma$ be a hyperbolic surface and let $\rho:\Gamma\to \mathsf{Homeo}(\mathbb{S}^1)$ be defined by extending the action of Γ on $\mathbb{H}^2$ to $\partial\mathbb{H}^2\approx\mathbb{S}^1$. The suspension

$$\mathcal{F}_\Gamma=\mathbb{H}^2\times_\rho\mathbb{S}^1$$

is called an *Anosov foliation*. Note that $\mathcal{F}_\Gamma$ is not an $\mathbb{S}^1$-suspension. It is classical that the underlying space of $\mathcal{F}_\Gamma$ is homeomorphic to the unit tangent bundle $\mathrm{T}^1_*\,\Sigma$.

2.2 Quasisuspensions

Let $\mathcal{L}_\rho=\widetilde{B}\times_\rho F$ be a suspension over a manifold B. We say that $\mathcal{L}_\rho$ is *Galois* if every leaf of $\mathcal{L}_\rho$ is Galois. Throughout this section, $\mathcal{L}_\rho$ will be a Galois suspension. For each leaf L pick a basepoint x_L lying over the basepoint of B. This allows us to define an action of $\pi_1 B$ on $\mathcal{L}_\rho$ by

$$x\;\longmapsto\;\bar{\gamma}\cdot x,$$

where, for x contained in the leaf L, $\bar{\gamma}$ is the image of $\gamma\in\pi_1 B$ in

$$\pi_1 B/(p_L)_*(\pi_1 L)\cong D_L=\text{ the deck group of } p|_L.$$

Let $\mathcal{X} \subset \mathcal{L}_\rho$ be any closed subset with $\mathcal{X} \cap L$ discrete for each leaf L, and which is invariant with respect to the action of $\pi_1 B$ (note that this does not depend on the choice of basepoints x_L). Let $\mathcal{L}_0 := \mathcal{L}_\rho \setminus \mathcal{X}$, which is a lamination mapping to B. If $\mathcal{L}_\rho$ is minimal, for any $x \in \mathcal{X}$ the orbit $\pi_1 B \cdot x$ is dense in the fiber F_x containing x, hence $F_x \subset \mathcal{X}$. It follows in this case that $\mathcal{X}$ is the union of fibers over a subset $X \subset B$ and $\mathcal{L}_0$ is a fiber bundle over $B_0 = B \setminus X$. In general, we shall define the fibers of $\mathcal{L}_0$ over $x \in B$ to be the preimages of the map $\mathcal{L}_0 \to B$.

A lamination homeomorphism $f : \mathcal{L}_0 \to \mathcal{L}_0$ is *weakly fiber-preserving* if for every fiber F_x over B,

$$f(F_x) = \bigcup_{i=1}^{n} E_{x_i}, \tag{1}$$

where E_y denotes a subset of the fiber F_y. The collection $\mathsf{Homeo}_{\omega-\mathsf{fib}}(\mathcal{L}_0)$ of weakly fiber-preserving homeomorphisms is clearly a group. Since the fibers are disjoint, each E_{x_i} occurring in (1) must be open in F_{x_i}. In particular, if the fibers are connected, a weakly fiber-preserving homeomorphism is fiber-preserving. Thus, the concept of a weakly fiber-preserving homeomorphism differs from that of a fiber-preserving homeomorphism when the fibers are disconnected *e.g.* when $\mathcal{L}_0$ is a solenoid.

Definition 1. *Let $\mathcal{L}_0$ be as above and suppose $H < \mathsf{Homeo}_{\omega-\mathsf{fib}}(\mathcal{L}_0)$ is a subgroup acting properly discontinuously on $\mathcal{L}_0$. The quotient*

$$\mathcal{Q} = H \backslash \mathcal{L}_0$$

is a lamination called a **quasisuspension** *(over B).*

We consider now two examples.

2.2.1 The Reeb foliation

Let $\mathbb{R}_+ = [0, \infty)$, consider the trivial suspension $\mathbb{C} \times \mathbb{R}_+$ over $\mathbb{C}$, and denote $(\mathbb{C} \times \mathbb{R}_+)^* = \mathbb{C} \times \mathbb{R}_+ \setminus \{(0, 0)\}$. (Thus we are taking $\mathcal{X} = \{(0, 0)\}$.) Fix $(\mu, \lambda) \in (\mathbb{C} \times \mathbb{R}_+)^*$ with $|\mu|, \lambda > 1$, $\mu \neq \lambda$. Then multiplication by (μ, λ) in $(\mathbb{C} \times \mathbb{R}_+)^*$ is a fiber-preserving lamination homeomorphism giving rise to an action by $\mathbb{Z}$. The resulting quasisuspension

$$\mathcal{F}_{\mathsf{Reeb}} = \mathbb{Z} \backslash (\mathbb{C} \times \mathbb{R}_+)^*$$

has underlying space a solid torus, and is called the *Reeb foliation*.

Let $P : (\mathbb{C} \times \mathbb{R}_+)^* \to \mathcal{F}_{\mathsf{Reeb}}$ denote the projection map. The leaves of $\mathcal{F}_{\mathsf{Reeb}}$ are of the form:

(1) $L_t = P(\mathbb{C} \times \{t\}) \cong \mathbb{C}$, for $t > 0$.

(2) $L_0 = P(\mathbb{C}^* \times \{0\}) \cong \mathbb{C}^*/ < \mu >$.

The fiber tranversals of $\mathcal{F}_{\mathsf{Reeb}}$ are of the form:

(1) $T_z = P(\{z\} \times \mathbb{R}_+) \approx \mathbb{R}_+$, $z > 0$. Every leaf of $\mathcal{F}_{\mathsf{Reeb}}$ intersects T_z.

(2) $T_0 = P(\{0\} \times (0, \infty)) \approx \mathbb{S}^1$. Every leaf except L_0 intersects T_0.

There is an action of $\mathbb{Z}$ on $\mathcal{F}_{\mathsf{Reeb}}$ induced by the map $(z,\ t) \mapsto (\mu^n z,\ t)$. For $x \in \mathcal{F}_{\mathsf{Reeb}}$, we write this action $x \mapsto n \cdot x$. For every t we have $n \cdot L_t = L_t$ and for all z, $n \cdot T_z = T_z$. Note that this action is the identity on L_0.

2.2.2 The Sullivan solenoid

The following important example comes from holomorphic dynamics. Let $U, V \subset \mathbb{C}$ be regions conformal to the unit disc, with $\overline{U} \subset V$. Recall that a *polynomial-like map* is a proper conformal map $f : U \to V$. The conjugacy class of f is uniquely determined by a pair $(p, \partial f)$, where p is a complex polynomial of degree d and $\partial f : \mathbb{S}^1 \to \mathbb{S}^1$ is a smooth, expanding map of degree d [3]. The space

$$\widehat{\mathbb{S}} \ = \ \lim_{\longleftarrow} \left(\mathbb{S}^1 \xleftarrow{\partial f} \mathbb{S}^1 \xleftarrow{\partial f} \mathbb{S}^1 \xleftarrow{\partial f} \cdots \right) \tag{2}$$

is an inverse limit solenoid which may be identified with the $\widehat{\mathbb{Z}}_d$-suspension $\mathcal{L}_\iota = \mathbb{R} \times_\iota \widehat{\mathbb{Z}}_d$, where $\widehat{\mathbb{Z}}_d$ is the group of d-adic integers and $\iota : \mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}_d$ is the canonical inclusion. Every leaf of $\widehat{\mathbb{S}}$ is homeomorphic to $\mathbb{R}$. ∂f defines a self map of the inverse system in (2), inducing a homeomorphism $\partial \hat{f} : \widehat{\mathbb{S}} \to \widehat{\mathbb{S}}$.

Consider the suspension

$$\widehat{\mathbb{D}} \ = \ \mathbb{H}^2 \times_\iota \widehat{\mathbb{Z}}_d$$

obtained by extending to $\mathbb{H}^2 \times \widehat{\mathbb{Z}}_d$ the identification used to define $\mathcal{L}_\iota$ *e.g.* $(z, \hat{n}) \sim (\gamma^m(z),\ \hat{n} - m)$ for $m \in \mathbb{Z}$, where $\gamma(z) = z + 1$. The base of the suspension $\widehat{\mathbb{D}}$ is the punctured hyperbolic disc $\mathbb{D}^* = \langle \gamma \rangle \backslash \mathbb{H}^2$, and its ideal boundary may be identified with $\widehat{\mathbb{S}}$.

The map $\partial \hat{f}$ extends to a weakly fiber-preserving homeomorphism $\hat{f} : \widehat{\mathbb{D}} \to \widehat{\mathbb{D}}$ which acts properly discontinuously on $\widehat{\mathbb{D}}$. The quotient

$$\widehat{\mathbb{D}}_f \ := \ \langle \hat{f} \rangle \backslash \widehat{\mathbb{D}}$$

is a quasisuspension called the *Sullivan solenoid* [13], [6].

2.3 Double coset foliations

Let $\mathfrak{G}$ be a Lie group, $\mathfrak{H}$ a closed Lie subgroup, $\Gamma < \mathfrak{G}$ a discrete subgroup. The foliation of $\mathfrak{G}$ by right cosets $\mathfrak{H}g$ descends to a foliation $\mathcal{F}_{\mathfrak{H},\Gamma}$ of $\mathfrak{G}/\Gamma$, called a *double coset foliation*.

For example, it is easy to see that if we take $\mathfrak{G} = \mathbb{R}^n$, $\mathfrak{H} = V$ a p-dimensional subspace and $\Gamma = \mathbb{Z}^n$, then the resulting double coset foliation is the linear foliation $\mathcal{F}_V$ of the torus $\mathbb{T}^n$.

Examples of double coset foliations which are not suspensions may be constructed as follows. Let Γ be a co-finite volume Fuchsian group. Denote by $\Sigma = \mathbb{H}^2/\Gamma$ and by $\mathrm{T}^1_* \Sigma$ the unit tangent bundle of Σ. Recall that every $v \in \mathrm{T}^1_* \mathbb{H}^2$ determines three oriented, parametrized curves: a geodesic γ and two horocycles $\mathfrak{h}_+$, $\mathfrak{h}_-$ tangent to, respectively, $\gamma(\infty)$ and $\gamma(-\infty)$. By parallel translating v along these curves, we obtain three flows on $\mathrm{T}^1_* \mathbb{H}^2$. The three flows are Γ-invariant, and define flows on $\mathrm{T}^1_* \Sigma$. The corresponding foliations are denoted Geod_Γ, Hor^+_Γ and Hor^-_Γ.

Now let $\mathfrak{G} = SL(2,\mathbb{R})$ and take $\mathfrak{H}$ to be one of the 1-parameter subgroups $H^+ = \{A^+_r\}$, $H^- = \{A^-_r\}$ and $G = \{B_r\}$, where

$$A^+_r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad A^-_r = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \text{ and } B_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}$$

for $r \in \mathbb{R}$. Then it is classical that the foliations $\mathcal{F}_{G,\Gamma}$ and $\mathcal{F}_{H^\pm,\Gamma}$ are homeomorphic to Geod_Γ and $\mathsf{Hor}^\pm_\Gamma$, respectively. Note also that the Anosov foliation $\mathcal{F}_\Gamma$ is homeomorphic to the sum $\mathsf{Geod}_\Gamma \oplus \mathsf{Hor}^+_\Gamma$.

2.4 Locally-free lie group actions

Let $\mathfrak{V}$ be a Lie group of dimension k, M^n an n-manifold, $n > k$, X a subspace of M^n. A continuous representation $\theta : \mathfrak{V} \to \mathsf{Homeo}(X)$ is called *locally free* if for all $x \in X$, the isotropy subgroup $I_x < \mathfrak{V}$ is discrete. If for any pair $x, y \in X$, their $\mathfrak{V}$-orbits are either disjoint or coincide, then X has the structure of a lamination $\mathcal{L}_{\mathfrak{V}}$ whose leaves are the $\mathfrak{V}$-orbits.

Once again, the linear foliation $\mathcal{F}_V$ fits into this framework: take $\mathfrak{V} = L_0$ = the leaf containing the identity, $M^n = X = \mathbb{T}^n$ and θ the map induced by addition in $\mathbb{T}^n$.

Here is an example which is neither a suspension nor a double coset. Let M^n be a Riemannian manifold. Fix a tangent vector $v \in \mathrm{T}_x M^n$. Let $l \subset M^n$ be the complete geodesic determined by v, X its closure (itself a union of geodesics). Then there is a locally free action of $\mathbb{R}$ given by geodesic flow along X, and X

is a lamination when l is simple. When $M^n = \Sigma$ is a hyperbolic surface and l is simple, we obtain a geodesic lamination in Σ in the sense of [14], a solenoid since its transversals are totally-disconnected.

3 The fundamental germ

Let $\mathcal{L}$ be any of the laminations considered in the previous section and let $L \subset \mathcal{L}$ be a fixed leaf. If $\mathcal{L} = H\backslash\mathcal{L}_0$ is a quasisuspension let $L_0 \subset \mathcal{L}_0$ be a leaf lying over L. The *diophantine group* G_L of $\mathcal{L}$ with respect to L is

- $\pi_1 B$ if $\mathcal{L}$ is a suspension.
- The group generated by $\pi_1 B$, $H_L = \{h \in H \mid h(L_0) = L_0\}$ and $\pi_1 L$ (viewed as groups acting on $\widetilde{B} \approx \widetilde{L}$) if $\mathcal{L}$ is a quasisuspension.
- The group $\widetilde{\mathfrak{H}}$ if $\mathcal{L}$ is a double coset.
- The group $\widetilde{\mathfrak{B}}$ if $\mathcal{L}$ is a locally free Lie group action.

Note that in every case, $\pi_1 L < G_L$.

Let $\hat{x} \in \mathcal{L}$ and T a transversal containing $\hat{x}$. Denote by $\widetilde{T}_L \subset \widetilde{L}$ the set of points lying over $T \cap L$. Then T is said to be a *diophantine transversal* if for every leaf L and $\tilde{x} \in \widetilde{T}_L$, any $\tilde{y} \in \widetilde{T}_L$ may be written in the form $\tilde{y} = g \cdot \tilde{x}$ for some $g \in G_L$. For $\tilde{x} \in \widetilde{T}_L$ fixed, we call $\{g_\alpha\} \subset G_L$ a G_L*-diophantine approximation* of $\hat{x}$ along T based at $\tilde{x}$ if $\{g_\alpha \cdot \tilde{x}\}$ projects in L to a sequence converging to $\hat{x}$ in T. The image of all such G_L-diophantine approximations in ${}^*G_L$ is denoted

$$ {}^*\mathsf{D}(\tilde{x}, \hat{x}, T), $$

and when $\tilde{x}$ projects to $\hat{x}$, we write ${}^*\mathsf{D}(\tilde{x}, T)$. If there are no G_L-diophantine approximations of $\hat{x}$ along T based at $\tilde{x}$, we define ${}^*\mathsf{D}(\tilde{x}, \hat{x}, T) = 0$. Note that if $\tilde{x}' = \gamma \cdot \tilde{x}$ for $\gamma \in \pi_1 L < G_L$ then

$$ {}^*\mathsf{D}(\tilde{x}', \hat{x}, T) \cdot \gamma \;=\; {}^*\mathsf{D}(\tilde{x}, \hat{x}, T). \tag{3} $$

Let ${}^*\mathsf{D}(\tilde{x}, \hat{x}, T)^{-1}$ consist of the set of inverses ${}^*g^{-1}$ of classes belonging to ${}^*\mathsf{D}(\tilde{x}, \hat{x}, T)$.

Definition 2. *Let $\mathcal{L}$, L, $\hat{x}$ and T be as above and let $x \in L \cap T$. The* **fundamental germ** *of $\mathcal{L}$ based at $\hat{x}$ along x and T is*

$$ [\![\pi]\!]_1(\mathcal{L}, x, \hat{x}, T) \;=\; {}^*\mathsf{D}(\tilde{x}, \hat{x}, T) \cdot {}^*\mathsf{D}(\tilde{x}, \hat{x}, T)^{-1} $$

where $\tilde{x}$ is any point in $\widetilde{L}$ lying over x.

By (3), $[\![\pi]\!]_1(\mathcal{L}, x, \hat{x}, T)$ does not depend on the choice of $\tilde{x}$ over x. When $x = \hat{x} \in L$, we write $[\![\pi]\!]_1(\mathcal{L}, x, T)$. Observe in this case that $[\![\pi]\!]_1(\mathcal{L}, x, T)$ contains a subgroup isomorphic to ${}^*\pi_1(L, x)$.

We now describe a groupoid structure on $[\![\pi]\!]_1(\mathcal{L}, x, \hat{x}, T)$. To do this, we define a unit space on which it acts: let ${}^\bullet\mathsf{D}(\tilde{x}, \hat{x}, T)$ be the image of ${}^*\mathsf{D}(\tilde{x}, \hat{x}, T)$ in ${}^\bullet G_L$, for any $\tilde{x}$ over x. We say that ${}^*u \in [\![\pi]\!]_1(\mathcal{L}, x, \hat{x}, T)$ is defined on ${}^\bullet g \in {}^\bullet\mathsf{D}(\tilde{x}, \hat{x}, T)$ if ${}^*u \cdot {}^\bullet g \in {}^\bullet\mathsf{D}(\tilde{x}, \hat{x}, T)$. Here we are using the left action of ${}^*G_L$ on ${}^\bullet G_L$. Having defined the domain and range of elements of $[\![\pi]\!]_1(\mathcal{L}, x, \hat{x}, T)$, it is easy to see that $[\![\pi]\!]_1(\mathcal{L}, x, \hat{x}, T)$ is a groupoid, as every element has an inverse by construction. This groupoid structure does not depend on the choice of $\tilde{x}$ over x.

4 The fundamental germ of a suspension

In the case of a suspension $\mathcal{L}_\rho = \widetilde{B} \times_\rho F$, any fiber over the base B is a diophantine transversal. Conversely, any diophantine transversal is an open subset of a fiber transversal. It follows that any two diophantine transversals T, T' through points $x, \hat{x}$ define the same set of G_L-diophantine approximations. Thus

Proposition 6. *If T and T' are diophantine transversals containing x and $\hat{x}$ then*

$$[\![\pi]\!]_1(\mathcal{L}_\rho, x, \hat{x}, T) \;=\; [\![\pi]\!]_1(\mathcal{L}_\rho, x, \hat{x}, T').$$

Accordingly for suspensions we drop mention of the transversal and write $[\![\pi]\!]_1(\mathcal{L}, x, \hat{x})$. We note that since the diophantine group $G_L = \pi_1 B$ is discrete, ${}^*G_L = {}^\bullet G_L$ and the unit space for the groupoid structure is just ${}^*\mathsf{D}(\tilde{x}, \hat{x})$.

4.1 Manifolds

A manifold is a lamination with just one leaf, which can be viewed as the suspension of the trivial representation of its fundamental group on a point. We have immediately

Proposition 7. *If M is a manifold then*

$$[\![\pi]\!]_1(M, x) \;=\; {}^*\mathsf{D}(\tilde{x}) \;=\; {}^*\pi_1(M, x).$$

4.2 $\mathfrak{G}$-suspensions

Let $\varphi : \pi_1 B \to \mathfrak{G}$ be a homomorphism, $\mathcal{L}_\varphi$ the corresponding $\mathfrak{G}$-suspension. Let $\{U_i\}$ be a neighborhood basis about 1 in $\mathfrak{G}$ and define a collection of nested sets $\{G_i\}$ by $G_i = \varphi^{-1}(U_i)$. Then the ultrascope $\bigodot G_i$ is a subgroup of ${}^*\pi_1 B$. In fact, if ${}^*\varphi : {}^*\pi_1 B \to {}^*\mathfrak{G}$ is the nonstandard version of φ, then

$$\bigodot G_i \;=\; {}^*\varphi^{-1}({}^*\mathfrak{G}_\epsilon).$$

Theorem 1. *For any pair $x, \hat{x}$ belonging to a diophantine transversal, $[\![\pi]\!]_1(\mathcal{L}_\varphi, x, \hat{x})$ is a group isomorphic to*

- $\bigodot G_i$ *if $\hat{x}$ belongs to the closure of the leaf containing x.*
- 0 *otherwise.*

Proof. Suppose that $\hat{x}$ belongs to the closure of the leaf containing x and let ${}^*g \in {}^*\mathsf{D}(\tilde{x}, \hat{x})$. Then any other element ${}^*g' \in {}^*\mathsf{D}(\tilde{x}, \hat{x})$ may be written in the form ${}^*g \cdot {}^*h$ where ${}^*h \in \bigodot G_i$. It follows immediately that

$$[\![\pi]\!]_1(\mathcal{L}_\varphi, x, \hat{x}) \;=\; {}^*g \cdot \left(\bigodot G_i\right) \cdot {}^*g^{-1} \;\cong\; \bigodot G_i.$$

Because the unit space ${}^*\mathsf{D}(\tilde{x}, \hat{x})$ is invariant under left-multiplication by any element of the fundamental germ, it follows that $[\![\pi]\!]_1(\mathcal{L}_\varphi, x, \hat{x})$ is a group, its composition law coinciding with multiplication in $\bigodot G_i$. If $\hat{x}$ does not belong the the closure of the leaf containing x, then ${}^*\mathsf{D}(\tilde{x}, \hat{x}) = 0$ by definition. □

For minimal $\mathfrak{G}$-suspensions we can thus reduce our notation to $[\![\pi]\!]_1(\mathcal{L}_\varphi)$.

Denote by ${}^*\pi_1 B_{\text{fin}}$ the subgroup ${}^*\varphi^{-1}({}^*\mathfrak{G}_{\text{fin}})$. The following theorem can be used to display many familiar topological groups as algebraic quotients of nonstandard versions of discrete groups.

Theorem 2. *If φ has dense image, then $[\![\pi]\!]_1(\mathcal{L}_\varphi)$ is a normal subgroup of ${}^*\pi_1 B_{\text{fin}}$ with*

$$ {}^*\pi_1 B_{\text{fin}} \big/ [\![\pi]\!]_1(\mathcal{L}_\varphi) \;\cong\; \mathfrak{G}.$$

Proof. Since φ has dense image, the composition of homomorphisms ${}^*\pi_1 B_{\text{fin}} \to {}^*\mathfrak{G}_{\text{fin}} \to \mathfrak{G}$ – where the first arrow is ${}^*\varphi$ – is surjective with kernel ${}^*\varphi^{-1}({}^*\mathfrak{G}_\epsilon) = [\![\pi]\!]_1(\mathcal{L}_\varphi)$. □

4.3 Inverse limit solenoids

Let $\widehat{M}$ be an inverse limit solenoid over the base M, and let $\{H_i\}$ be a sequence of subgroups of $\pi_1 M$ cofinal in the collection of subgroups in the defining inverse system. By the discussion in §2.1.1, the collection of closures $\{\widehat{H}_i\} \subset \hat{\pi}_1 M$

defines a neighborhood basis about 1. Since $\widehat{M}$ is a $\hat{\pi}_1 M$-suspension in which φ is dense, it follows from Theorem 1 that $[\![\pi]\!]_1(\widehat{M}, x, \hat{x})$ is a group isomorphic to $\bigodot H_i$.

For example, consider a solenoid $\widehat{\mathbb{S}}$ over $\mathbb{S}^1$. Here, each H_i is an ideal in $\mathbb{Z}$, hence $[\![\pi]\!]_1\widehat{\mathbb{S}}$ is an ideal in the ring ${}^*\mathbb{Z}$ = nonstandard $\mathbb{Z}$. When $H_i = (d^i)$ for $d \in \mathbb{Z}$ fixed, we denote the resulting germ ${}^*\mathbb{Z}_{\hat{\epsilon}}(d)$ and when $H_i = (i)$ we write ${}^*\mathbb{Z}_{\hat{\epsilon}}$. Being uncountable, these ideals are not principal, so ${}^*\mathbb{Z}$, unlike $\mathbb{Z}$, is not a PID. By Theorem 2, we have ${}^*\mathbb{Z}/{}^*\mathbb{Z}_{\hat{\epsilon}} \cong \widehat{\mathbb{Z}}$ and ${}^*\mathbb{Z}/{}^*\mathbb{Z}_{\hat{\epsilon}}(d) \cong \widehat{\mathbb{Z}}_d$.

4.4 Linear foliations of torii and classical diophantine approximation

Let $\mathcal{F}_V$ be the linear foliation of $\mathbb{T}^n$ associated to the subspace $V \subset \mathbb{R}^n$. As in § 2.1.2, we regard V as the graph of the $q \times p$ matrix $\mathbf{R}$. Let $\varphi_{\mathbf{R}} : \mathbb{Z}^p \to \mathbb{T}^q$ be the homomorphism used to define $\mathcal{F}_V$. Let $\{U_i\}$ be a neighborhood basis in $\mathbb{T}^q$ about $\bar{\mathbf{0}}$. We define a nested set $\{G_i\} \subset \mathbb{Z}^p$ by $G_i = \varphi_{\mathbf{R}}^{-1}(U_i)$. Denote

$$ {}^*\mathbb{Z}^p_{\mathbf{R}} := \bigodot G_i = {}^*\varphi_{\mathbf{R}}^{-1}({}^*\mathbb{T}^q_{\epsilon}), $$

a subgroup of ${}^*\mathbb{Z}^p$. If $p = q = 1$ and $\mathbf{R} = r \in \mathbb{R}$, we write instead ${}^*\mathbb{Z}_r$. Since $\mathcal{F}_V$ is a $\mathbb{T}^q$-foliation, we have by Theorem 1 that $[\![\pi]\!]_1(\mathcal{F}_V, x, \hat{x}) = {}^*\mathbb{Z}^p_{\mathbf{R}}$ when $\hat{x}$ belongs to the closure of the leaf containing x, and is 0 otherwise. Applying Theorem 2 we have that every finite dimensional torus $\mathbb{T}^q$ is algebraically isomorphic to a quotient of ${}^*\mathbb{Z}$.

Theorem 3. *${}^*\mathbb{Z}^p_{\mathbf{R}}$ is an ideal in ${}^*\mathbb{Z}^p$ if and only if $\mathbf{R} \in M_{q,p}(\mathbb{Q})$.*

Proof. Suppose that $\mathbf{R} \in M_{q,p}(\mathbb{Q})$ and let a_k = the l.c.d. of the entries of $\mathbf{r}_k$ = the kth column of $\mathbf{R}$. Write

$$ \mathfrak{a} = (a_1) \oplus \cdots \oplus (a_p) $$

where (a_k) is the ideal generated by a_k. Note that ${}^*\mathfrak{a} \subset {}^*\mathbb{Z}^p_{\mathbf{R}}$. On the other hand, rationality of the entries of the $\mathbf{r}_k$ implies that a sequence $\{\mathbf{n}_\alpha\} \subset \mathbb{Z}^p$ defines an element of ${}^*\mathbb{Z}^p_{\mathbf{R}}$ if and only if there exists $X \in \mathfrak{U}$ such that $\varphi_{\mathbf{R}}(\mathbf{n}_\alpha) = \bar{\mathbf{0}}$ for all $\alpha \in X$. This is equivalent to $\mathbf{n}_\alpha \in \mathfrak{a}$ for all $\alpha \in X$. Thus ${}^*\mathbb{Z}^p_{\mathbf{R}} = {}^*\mathfrak{a}$ which is an ideal in ${}^*\mathbb{Z}^p$.

Suppose now that $\mathbf{r} = \mathbf{r}_k \notin \mathbb{Q}^q$ for some k, $1 \leq k \leq p$. Let $\{\mathbf{n}_\alpha\}$ represent an element ${}^*\mathbf{n} \in {}^*\mathbb{Z}^p_{\mathbf{R}}$, and denote by $\{n_\alpha\}$ the sequence of k-th coordinates of the $\mathbf{n}_\alpha$. Note that $\overline{n_\alpha \mathbf{r}} \neq \bar{\mathbf{0}}$ for all α since $\mathbf{r}$ is not rational. In fact, for any $\delta > 0$ we may find a sequence of integers $\{m_\alpha\}$ such that $\overline{m_\alpha n_\alpha \mathbf{r}}$ is *not* within δ of $\bar{\mathbf{0}}$. Let $\mathbf{m}_\alpha \in \mathbb{Z}^p$ be the vector whose kth coordinate is m_α and whose other coordinates are 0. Then the sequence $\{\mathbf{m}_\alpha \cdot \mathbf{n}_\alpha\}$ does not converge with respect to $\{G_i\}$ *i.e.* ${}^*\mathbf{m} \cdot {}^*\mathbf{n} \notin {}^*\mathbb{Z}^p_{\mathbf{R}}$, so ${}^*\mathbb{Z}^p_{\mathbf{R}}$ is not an ideal. □

Theorem 3 draws another sharp distinction between $\mathbb{Z}$ and ${}^*\mathbb{Z}$: every subgroup of the former is an ideal, while this is false for the latter.

We spend the rest of this section studying ${}^*\mathbb{Z}^p_{\mathbf{R}}$, in and of itself a complicated and intriguing object. Let us begin with the following alternate description of ${}^*\mathbb{Z}^p_{\mathbf{R}}$:

$$ {}^*\mathbb{Z}^p_{\mathbf{R}} = \left\{ {}^*\mathbf{n} \in {}^*\mathbb{Z}^p \;\middle|\; \exists\, {}^*\mathbf{n}^\perp \in {}^*\mathbb{Z}^q \text{ such that } \mathbf{R}({}^*\mathbf{n}) - {}^*\mathbf{n}^\perp \in {}^*\mathbb{R}^q_\epsilon \right\}. \tag{4} $$

Given ${}^*\mathbf{n} \in {}^*\mathbb{Z}^p_{\mathbf{R}}$, the corresponding element ${}^*\mathbf{n}^\perp \in {}^*\mathbb{Z}^q$ is called the *dual* of ${}^*\mathbf{n}$; it is uniquely determined. From (4), it is clear that the set

$$ ({}^*\mathbb{Z}^p_{\mathbf{R}})^\perp := \left\{ {}^*\mathbf{n}^\perp \;\middle|\; {}^*\mathbf{n}^\perp \text{ is the dual of } {}^*\mathbf{n} \in {}^*\mathbb{Z}^p_{\mathbf{R}} \right\} $$

is a subgroup of ${}^*\mathbb{Z}^q$, called the *dual* of ${}^*\mathbb{Z}^p_{\mathbf{R}}$. Note that when $\mathbf{R} \in M_{q,p}(\mathbb{R} \setminus \mathbb{Q})$ has a left-inverse $\mathbf{S}$, we have $({}^*\mathbb{Z}^p_{\mathbf{R}})^\perp = {}^*\mathbb{Z}^q_{\mathbf{S}}$.

Similarly, the set

$$ {}^*\mathbb{R}^q_{\mathbf{R},\epsilon} = \left\{ {}^*\epsilon \in {}^*\mathbb{R}^q_\epsilon \;\middle|\; \exists {}^*\mathbf{n} \in {}^*\mathbb{Z}^p_{\mathbf{R}} \text{ such that } \mathbf{R}({}^*\mathbf{n}) - {}^*\mathbf{n}^\perp = {}^*\epsilon \right\} $$

is a subgroup of ${}^*\mathbb{R}^q_\epsilon$, called the *group of rates* of $\mathbf{R}$.

The following proposition is an immediate consequence of (4).

Proposition 8. *The maps* ${}^*\mathbf{n} \mapsto {}^*\mathbf{n}^\perp$ *and* ${}^*\mathbf{n} \mapsto {}^*\epsilon$ *define isomorphisms*

$$ {}^*\mathbb{Z}^p_{\mathbf{R}} \cong ({}^*\mathbb{Z}^p_{\mathbf{R}})^\perp \quad \textit{and} \quad {}^*\mathbb{Z}^p_{\mathbf{R}} \cong {}^*\mathbb{R}^q_{\mathbf{R},\epsilon}. $$

Note 1 (A.Verjovsky). Using formulation (4) of ${}^*\mathbb{Z}^p_{\mathbf{R}}$, it follows that every triple

$$ ({}^*\mathbf{n},\ {}^*\mathbf{n}^\perp,\ {}^*\epsilon) $$

represents a diophantine approximation of $\mathbf{R}$. Thus we may regard ${}^*\mathbb{Z}^p_{\mathbf{R}}$ as the *group of diophantine approximations* of $\mathbf{R}$.

For example, when $p = q = 1$ and $r \in \mathbb{R} \setminus \mathbb{Q}$, ${}^*n$ and ${}^*n^\perp$ are equivalence classes of sequences $\{x_\alpha\}$ and $\{y_\alpha\} \subset \mathbb{Z}$, and ${}^*\epsilon$ an equivalence class of sequence $\{\epsilon_\alpha\} \subset \mathbb{R}$, $\epsilon_\alpha \to 0$, such that

$$ \left| r - \frac{y_\alpha}{x_\alpha} \right| = \left| \frac{\epsilon_\alpha}{x_\alpha} \right| \longrightarrow 0. $$

Conversely, every diophantine approximation of r defines uniquely a triple

$$ ({}^*n, {}^*n^\perp, {}^*\epsilon). $$

Recall that two irrational numbers $r, s \in \mathbb{R} \setminus \mathbb{Q}$ are *equivalent* if there exists

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

such that $s = A(r) = (ar + b)/(cr + d)$.

Proposition 9. *If r and s are equivalent irrational numbers, then ${}^*\mathbb{Z}_r \cong {}^*\mathbb{Z}_s$.*

Proof. Given ${}^*n \in {}^*\mathbb{Z}_r$, observe that $(cr + d){}^*n \simeq c{}^*n^\perp + d{}^*n \in {}^*\mathbb{Z}$. Write ${}^*m = c{}^*n^\perp + d{}^*n$. Then ${}^*m \in {}^*\mathbb{Z}_s$, since

$$s{}^*m \simeq (ar + b){}^*n \simeq a{}^*n^\perp + b{}^*n \in {}^*\mathbb{Z}.$$

The association ${}^*n \mapsto {}^*m$ defines an injective homomorphism $\psi : {}^*\mathbb{Z}_r \to {}^*\mathbb{Z}_s$, with inverse defined $\psi^{-1}({}^*m) \simeq (-cs + a){}^*m$. □

Note 2. Two irrational numbers r, s are called *virtually equivalent* if there exists $A \in SL(2, \mathbb{Q})$ such that $A(r) = s$. In this case, there exists a pair of monomorphisms

$$\psi_1 : {}^*\mathbb{Z}_r \hookrightarrow {}^*\mathbb{Z}_s \quad \text{and} \quad \psi_2 : {}^*\mathbb{Z}_s \hookrightarrow {}^*\mathbb{Z}_r,$$

defined as in Proposition 9. In other words, ${}^*\mathbb{Z}_r$ and ${}^*\mathbb{Z}_s$ are *virtually isomorphic*. These maps are mutually inverse to each other if and only if $A \in SL(2, \mathbb{Z})$.

We are led to make the following conjecture.

Conjecture 1. *If ${}^*\mathbb{Z}_r \cong {}^*\mathbb{Z}_s$ for irrational numbers r, s, then r and s are equivalent.*

A verified Conjecture 1 would augur a group theoretic approach to diophantine approximation.

4.5 Anosov foliations and hyperbolic diophantine approximation

Let Γ be a discrete subgroup of $PSL(2, \mathbb{R})$ with no elliptics, $\Sigma = \Gamma \backslash \mathbb{H}^2$ the corresponding Riemann surface. Let $\rho : \Gamma \to \mathsf{Homeo}(\mathbb{S}^1)$ be the representation of Γ on $\mathbb{S}^1 \approx \partial\mathbb{H}^2$ and denote as in § 2.1.3 the associated Anosov foliation by $\mathcal{F}_\Gamma$. Fix $t, \xi \in \mathbb{S}^1$, consider a neighborhood basis $\{U_i(\xi)\}$ about ξ, and define the nested set $\{G_i(t; \xi)\} \subset \Gamma$ by

$$G_i(t; \xi) = \left\{ A \in \Gamma \;\middle|\; \rho_A(t) \in U_i(\xi) \right\}.$$

Theorem 4. *Let $\hat{x} \in \mathcal{F}_\Gamma$ be contained in a leaf covered by $\mathbb{H} \times \{\xi\}$ and let x be contained in a leaf covered by by $\mathbb{H} \times \{t\}$. Then*

$$[\![\pi]\!]_1(\mathcal{F}, x, \hat{x}) \;=\; \bigodot \big(G_i(t;\xi) \cdot G_i(t;\xi)^{-1}\big)$$

if $\hat{x}$ is contained in the closure of the leaf containing x, and is 0 *otherwise.*

Proof. Immediate from the definition of $[\![\pi]\!]_1$. □

Classically [11], given $\xi \in \mathbb{S}^1$ in the limit set of Γ and $t \in \mathbb{S}^1$, a Γ-*hyperbolic diophantine approximation* of ξ based at t is a sequence $\{A_\alpha\} \subset \Gamma$ such that $|\xi - A_\alpha(t)| \to 0$, where $|\cdot|$ is the norm induced by the inclusion $\mathbb{S}^1 \subset \mathbb{R}^2$. It follows from our definitions that ${}^*\mathsf{D}(\tilde{x}, \hat{x})$ consists precisely of equivalence classes of Γ-hyperbolic diophantine approximations.

5 The fundamental germ of a quasisuspension

Let $\mathcal{L}_\rho$ be a Galois suspension, $\mathcal{X} \subset \mathcal{L}_\rho$ a $\pi_1 B$ invariant closed set, $\mathcal{L}_0 = \mathcal{L}_\rho \setminus \mathcal{X}$. Let $H < \mathsf{Homeo}_{\omega-\mathsf{fib}}(\mathcal{L}_0)$ be a subgroup acting properly discontinuously and let $\mathcal{Q} = H \backslash \mathcal{L}_0$ be the resulting quasisuspension. See §2.2. We have the following analogue of Proposition 6:

Proposition 10. *If T and T' are diophantine transversals containing x and $\hat{x}$ then*

$$[\![\pi]\!]_1(\mathcal{Q}, x, \hat{x}, T) \;=\; [\![\pi]\!]_1(\mathcal{Q}, x, \hat{x}, T').$$

Proof. The transversals T and T' lift to $\pi_1 B$ transversals in $\mathcal{L}_0$, which by Proposition 6 yield equivalent sets of $\pi_1 B$-diophantine approximations. This implies that ${}^*\mathsf{D}(\tilde{x}, \hat{x}, T) = {}^*\mathsf{D}(\tilde{x}, \hat{x}, T')$. □

Accordingly, we drop mention of T and write $[\![\pi]\!]_1(\mathcal{Q}, x, \hat{x})$.

Note 3. Note that ${}^*\pi_1(L)$ is a subgroup of $[\![\pi]\!]_1(\mathcal{Q}, x, \hat{x})$. In addition, there is a monomorphism

$$[\![\pi]\!]_1(\mathcal{L}_\rho, x, \hat{x}) \hookrightarrow [\![\pi]\!]_1(\mathcal{Q}, x, \hat{x}),$$

an isomorphism if $H_L = \{1\}$ and $\mathcal{X} = \emptyset$.

5.1 The Reeb foliation

Let $[\![\mathbb{Z}]\!]$ be the groupoid whose morphisms are elements of ${}^*\mathbb{Z}$, with the composition ${}^*m \circ {}^*n := {}^*m + {}^*n$ defined if and only if ${}^*m + {}^*n = 0 \mod \mathbb{Z}$. Recall that L_0 is the torus leaf.

Theorem 5. *For any pair* $x, \hat{x} \in \mathcal{F}_{\mathsf{Reeb}}$ *contained in a diophantine transversal with* $x \in L$,

$$[\![\pi]\!]_1(\mathcal{F}_{\mathsf{Reeb}}, x, \hat{x}) \cong \begin{cases} {}^*\mathbb{Z}^2 & \text{if } x = \hat{x} \in L_0 = L \\ [\![\mathbb{Z}]\!] & \text{if } \hat{x} \in L_0 \neq L \\ 0 & \text{otherwise} \end{cases}$$

Proof. Suppose first that $x = \hat{x} \in L_0$. Then $[\![\pi]\!]_1(\mathcal{F}_{\mathsf{Reeb}}, x, \hat{x}) = {}^*\pi_1 L_0 = {}^*\mathbb{Z}^2$. Now if $x \in L$, $\hat{x} \in L_0 \neq L$ are contained in a diophantine transversal, then a sequence $\{n_\alpha\} \subset \mathbb{Z}$ is a diophantine approximation if and only if it is infinite. Thus ${}^*\mathsf{D}(\tilde{x}, \hat{x}) = {}^*\mathbb{Z}_\infty := {}^*\mathbb{Z} \setminus {}^*\mathbb{Z}_{\text{fin}}$, the infinite nonstandard integers. Then as a set

$$[\![\pi]\!]_1(\mathcal{F}_{\mathsf{Reeb}}, x, \hat{x}) = {}^*\mathbb{Z}_\infty - {}^*\mathbb{Z}_\infty = {}^*\mathbb{Z}.$$

The domain of an element ${}^*n$ consists of those ${}^*a \in {}^*\mathbb{Z}_\infty$ for which ${}^*n + {}^*a \in {}^*\mathbb{Z}_\infty$. It is then clear that the law of composition is that of the groupoid $[\![\mathbb{Z}]\!]$. If $x \neq \hat{x}$ and $L = L_0$, there is no diophantine transversal containing the two points hence the fundamental germ is 0. If $\hat{x} \in L' \neq L_0$, there are no accumulations of L on L' so the fundamental germ is 0. □

Intuitively, when $\hat{x} \in L_0 \neq L$, $[\![\pi]\!]_1(\mathcal{F}_{\mathsf{Reeb}}, x, \hat{x})$ records the approximation by L of the circumferential cycle $c \subset L_0$ through $\hat{x}$. On the other hand, $[\![\pi]\!]_1(\mathcal{F}_{\mathsf{Reeb}}, x, \hat{x})$ does not predict the meridian cycle $c' \subset L_0$. Instead, c' is approximated by a sequence of inessential loops in L that move off to infinity, and such sequences are not the stuff of $[\![\pi]\!]_1$.

5.2 The Sullivan solenoid

Consider the Baumslag-Solitar group

$$G_{\mathsf{BS}} = G_{\mathsf{BS}}(d) = \left\langle f, x \ : \ fxf^{-1} = x^d \right\rangle.$$

Observe by induction that

$$x^{rd^\alpha} f = f x^{rd^{\alpha-1}} \quad \text{and} \quad f^{-1} x^{rd^\alpha} = x^{rd^{\alpha-1}} f^{-1} \tag{5}$$

for all $\alpha > 0$ and $r \in \mathbb{Z}$. Define a nested set about 1 by

$$G_i = \left\{ f^m x^{rd^i} \;\middle|\; m, r \in \mathbb{Z} \right\}. \tag{6}$$

Theorem 6. $\bigodot(G_i \cdot G_i^{-1})$ *is a group.*

Proof. To see that $\bigodot(G_i \cdot G_i^{-1})$ is a group, it suffices to check that $G_i \cdot G_i^{-1}$ is a group for all i. Write a generic element $g \in G_i \cdot G_i^{-1}$ in the form $g = f^l x^{rd^i} f^m$ for $l, m, r \in \mathbb{Z}$. Then an element gh^{-1}, $g, h \in G_i \cdot G_i^{-1}$ may be written (using (5))

$$gh^{-1} = f^l x^{rd^i} f^m x^{sd^i} f^n = \begin{cases} f^l x^{(r+sd^m)d^i} f^{m+n} & \text{if } m > 0 \\ f^{l+m} x^{(rd^m+s)d^i} f^n & \text{if } m \leq 0 \end{cases},$$

where $l, m, n, r, s \in \mathbb{Z}$. It follows that $gh^{-1} \in G_i \cdot G_i^{-1}$. □

Note 4. The ultrascope $\bigodot G_i$ is not even a groupoid as elements do not have inverses. Indeed, consider the sequence $\{g_\alpha\} = \left\{ f^{-m_\alpha} x^{d^\alpha} \right\}$, where $m_\alpha > \alpha > 0$, $\alpha = 1, 2, \ldots$. Note that $\{g_\alpha\}$ defines an element of $\bigodot G_i$. Using (5), we may write the inverse sequence

$$\{g_\alpha^{-1}\} = \left\{ x^{-d^\alpha} f^{m_\alpha} \right\} = \left\{ f^\alpha x^{-1} f^{m_\alpha - \alpha} \right\}.$$

Since $m_\alpha > \alpha$, we cannot use the defining relation of G_{BS} to move the remaining $f^{m_\alpha - \alpha}$ to the left of the x-term. It follows that $\{g_\alpha^{-1}\}$ does not define an element of $\bigodot G_i$, so the latter does not have the structure of a groupoid.

Let $[\![G_{\mathsf{BS}}]\!]$ be the groupoid defined by the action of $\bigodot(G_i \cdot G_i^{-1})$ on $\bigodot G_i$. Thus, we define the domain of ${}^*g \in \bigodot(G_i \cdot G_i^{-1})$ to be the set of ${}^*a \in \bigodot G_i$ for which ${}^*g \cdot {}^*a \in \bigodot G_i$, where $\cdot$ is multiplication in ${}^*G_{\mathsf{BS}}$.

Theorem 7. *For all* $x, \hat{x} \in \mathbb{D}_f$ *with* $x \in L$,

$$[\![\pi]\!]_1(\widehat{\mathbb{D}}_f, x, \hat{x}) \cong \begin{cases} [\![G_{\mathsf{BS}}]\!] & \textit{if } L \textit{ is an annulus} \\ {}^*\mathbb{Z}_{\hat{\epsilon}}(d) & \textit{if } L \textit{ is a disk} \end{cases}$$

Proof. First suppose L is an annulus and that $x = \hat{x}$. The action of $\pi_1\mathbb{D}^* \cong \mathbb{Z}$ on $\widehat{\mathbb{D}}$ is generated by $(z, \hat{n}) \mapsto (z, \hat{n} + 1)$, where $(z, \hat{n}) \in \mathbb{H}^2 \times \widehat{\mathbb{Z}}_d$. Then if γ is the generator of $\pi_1\mathbb{D}^*$, we have $\hat{f}\gamma \hat{f}^{-1} = \gamma^d$. It follows that the diophantine group is isomorphic to G_{BS}. The set of diophantine approximations ${}^*\mathsf{D}(\tilde{x}, \hat{x})$ is equal to $\bigodot G_i$, where G_i is the nested set (6). The result now follows by definition of $[\![\pi]\!]_1$. If $\hat{x} \neq x$, let $\hat{x}_0, x_0$ be lifts to $\widehat{\mathbb{D}}$ contained in a diophantine transversal and let $\{\gamma^{s_i}\}$ be a diophantine approximation of $\hat{x}_0$ based at x_0. Then ${}^*\mathsf{D}(\tilde{x}, \hat{x}) = \bigodot(G_i \cdot \gamma^{s_i})$ and therefore $[\![\pi]\!]_1(\widehat{\mathbb{D}}_f, x, \hat{x})$ is once again the groupoid $[\![G_{\mathsf{BS}}]\!]$. Now suppose that L is a disk: then by Note 3, $[\![\pi]\!]_1(\widehat{\mathbb{D}}_f, x, \hat{x}) = [\![\pi]\!]_1(\widehat{\mathbb{D}}, x_0, \hat{x}_0)$ where $(x_0, \hat{x}_0)$ covers $(x, \hat{x})$. By the results of § 4.3 we have $[\![\pi]\!]_1(\widehat{\mathbb{D}}, x_0, \hat{x}_0) = {}^*\mathbb{Z}_{\hat{\epsilon}}(d)$. □

The example of $\widehat{\mathbb{D}}_f$ illustrates the advantage of the "nonabelian Grothendieck group" type construction used in Definition 2: by Note 4, the naive choice "$[\![\pi]\!]_1 = {}^*\mathsf{D}$" would not even have produced a groupoid.

We now describe the groupoid structure of $[\![G_{\mathsf{BS}}]\!]$. Given ${}^*n \in {}^*\mathbb{Z}$ and $y \in G_{\mathsf{BS}}$, let $y^{{}^*n} \in {}^*G_{\mathsf{BS}}$ denote the sequence class of $\{y^{n_\alpha}\}$, where $\{n_\alpha\}$ represents ${}^*n$. For ${}^*g \in [\![G_{\mathsf{BS}}]\!]$, we may write

$$ {}^*g = f^{{}^*l} x^{{}^*r d^{{}^*u}} f^{{}^*m} $$

where ${}^*l, {}^*m, {}^*r \in {}^*\mathbb{Z}$ and ${}^*u \in {}^*\mathbb{N}_\infty$ = the infinite nonstandard naturals. We assume here that d does not divide ${}^*r$, so that the exponent ${}^*u$ is optimal. Define the *left and right degrees* of ${}^*g$ by

$$ \mathtt{ldeg}({}^*g) = {}^*l + {}^*u \quad \text{and} \quad \mathtt{rdeg}({}^*g) = {}^*u - {}^*m. $$

We note that the left and right degrees are invariant with respect to moving factors of f to the left or right of the x term using (5). In fact, we can always write

$$ {}^*g = f^{\mathtt{ldeg}({}^*g)} x^{{}^*r} f^{-\mathtt{rdeg}({}^*g)}. $$

For ${}^*a = f^{{}^*n} x^{{}^*s d^{{}^*v}} \in \bigodot G_i$, define the *degree* as

$$ \mathtt{deg}({}^*a) = {}^*n. $$

By (5) and the definition of $\bigodot G_i$ it follows that

$$ \mathtt{Dom}({}^*g) = \{{}^*a |\ \mathtt{rdeg}({}^*g) - \mathtt{deg}({}^*a) \in {}^*\mathbb{N}_\infty\}. $$

Indeed we have

$$ {}^*g \cdot {}^*a = f^{(\mathtt{ldeg}({}^*g) - \mathtt{rdeg}({}^*g) + \mathtt{deg}({}^*a))} x^{({}^*r d^{\mathtt{rdeg}({}^*g) - \mathtt{deg}({}^*a)} + {}^*s d^{{}^*v})} \tag{7} $$

if and only if $\mathtt{rdeg}({}^*g) - \mathtt{deg}({}^*a) \geq 0$, and in this event, the right hand side of (7) belongs to $\bigodot G_i$ if and only if $\mathtt{rdeg}({}^*g) - \mathtt{deg}({}^*a)$ is infinite.

Now let ${}^*h$ be another element of $[\![G_{\mathsf{BS}}]\!]$.

Theorem 8. *The composition ${}^*h \circ {}^*g$ is defined if and only if*

$$ \mathtt{rdeg}({}^*h) = \mathtt{ldeg}({}^*g) \mod \mathbb{Z}, \tag{8} $$

and in this event, is equal to ${}^*h \cdot {}^*g$.

Proof. Assume condition (8) and write $c = \mathtt{rdeg}({}^*h) - \mathtt{ldeg}({}^*g) \in \mathbb{Z}$. Let ${}^*a$ be in the domain of ${}^*g$. Then by (7), ${}^*g \cdot {}^*a$ is in the domain of ${}^*h$ if and only if

$$\mathtt{rdeg}({}^*h) - (\mathtt{ldeg}({}^*g) - \mathtt{rdeg}({}^*g) + \mathtt{deg}({}^*a)) = \mathtt{rdeg}({}^*g) - \mathtt{deg}({}^*a) + c$$

belongs to ${}^*\mathbb{N}_\infty$, which is true since ${}^*a \in \mathsf{Dom}({}^*g)$. On the other hand, suppose that ${}^*b = f^{\mathtt{deg}({}^*b)} x^{{}^*s' d^{{}^*v'}} \in \mathsf{Dom}({}^*h)$. Let ${}^*a = f^{\mathtt{deg}({}^*a)} x^{{}^*s d^{{}^*v}}$ where

$$\mathtt{deg}({}^*a) = \mathtt{deg}({}^*b) - \mathtt{ldeg}({}^*g) + \mathtt{rdeg}({}^*g)$$

and where ${}^*v$ is defined according to the following cases:

- If ${}^*v' \leq \mathtt{ldeg}({}^*g) - \mathtt{deg}({}^*b)$, take

$$ {}^*v = {}^*v' \quad \text{and} \quad {}^*s = {}^*s' - {}^*r d^{\mathtt{ldeg}({}^*g) - \mathtt{deg}({}^*b) - {}^*v'}.$$

- If ${}^*v' \geq \mathtt{ldeg}({}^*g) - \mathtt{deg}({}^*b)$, take

$$ {}^*v = \mathtt{ldeg}({}^*g) - \mathtt{deg}({}^*b) \quad \text{and} \quad {}^*s = {}^*s' d^{{}^*v' - (\mathtt{ldeg}({}^*g) - \mathtt{deg}({}^*b))} - {}^*r.$$

It follows that ${}^*a$ is in the domain of ${}^*g$ and ${}^*g \cdot {}^*a = {}^*b$. Thus $\mathsf{Ran}({}^*g) = \mathsf{Dom}({}^*h)$, so that the composition is defined. One can easily check that (8) implies that $\mathsf{Dom}({}^*h \cdot {}^*g) = \mathsf{Dom}({}^*g)$, so that this composition coincides with ${}^*h \cdot {}^*g$ as a morphism. Now assume that condition (8) does not hold so that ${}^*c = \mathtt{rdeg}({}^*h) - \mathtt{ldeg}({}^*g) \in {}^*\mathbb{Z}_\infty$. If ${}^*c < 0$, choose ${}^*a$ in the domain of ${}^*g$ such that

$$\mathtt{rdeg}({}^*g) - \mathtt{deg}({}^*a) = -{}^*c.$$

Then ${}^*g \cdot {}^*a$ is not in the domain of ${}^*h$. Likewise, if ${}^*c > 0$, pick ${}^*b \in \mathsf{Dom}({}^*h)$ so that $\mathtt{rdeg}({}^*h) - \mathtt{deg}({}^*b) = {}^*c$. Then ${}^*b$ is in the domain of ${}^*h$ but not in the range of ${}^*g$. □

Note 5. Let ${}^*\mathbb{Z}^2_{\mathsf{jux}}$ be the groupoid whose morphisms consist of pairs $({}^*m, {}^*n) \in {}^*\mathbb{Z}^2$ and an identity 1, and where the composition $({}^*m', {}^*n') \circ ({}^*m, {}^*n)$ is defined when ${}^*m - {}^*n' = c \in \mathbb{Z}$, equal to $({}^*m', {}^*n - c)$ or $({}^*m' - c, {}^*n)$ depending on whether c is positive or negative. Then the association ${}^*g \mapsto (\mathtt{ldeg}({}^*g), \mathtt{rdeg}({}^*g))$ if ${}^*g \neq 1$ and $1 \mapsto 1$ defines a surjective groupoid homomorphism $[\![G_{\mathsf{BS}}]\!] \to {}^*\mathbb{Z}^2_{\mathsf{jux}}$.

6 The fundamental germ of a double coset foliation

Let $\mathfrak{G}$ be a Lie group, $\mathfrak{H} < \mathfrak{G}$ a closed Lie subgroup, $\Gamma < \mathfrak{G}$ a discrete subgroup and $\mathcal{F}_{\mathfrak{H},\Gamma}$ the associated double coset foliation. The situation is considerably more subtle here due to the fact that the diophantine group is no longer discrete. Thus two choices of diophantine transversal T_1, T_2 through $x, \hat{x}$ yield distinct sets of diophantine approximations, in contrast with the case of a (quasi)suspension. Note on the other hand that *every* transversal is diophantine, since the universal covers of the leaves are homogeneous with respect to the left action of the diophantine group $\widetilde{\mathfrak{H}}$. In fact, if x_1 and x_2 are contained in the same leaf, then $\tilde{a} \cdot \tilde{x}_1 = \tilde{x}_2$ for some $\tilde{a} \in \widetilde{\mathfrak{H}}$. This yields a bijection of diophantine sets

$$
{}^*\mathsf{D}(\tilde{x}_1, \hat{x}, T_1) \longrightarrow {}^*\mathsf{D}(\tilde{x}_2, \hat{x}, T_2)
$$

defined ${}^*g_1 \mapsto {}^*g_2$ if ${}^\bullet g_1 = {}^\bullet g_2 \cdot \tilde{a}$ in ${}^\bullet\widetilde{\mathfrak{H}}$. That is, the bijection is given by the equality ${}^\bullet\mathsf{D}(\tilde{x}_1, \hat{x}, T_1) = {}^\bullet\mathsf{D}(\tilde{x}_2, \hat{x}, T_2) \cdot \tilde{a}$. However, it is not clear that the following prescription for a map of fundamental germs:

$$
{}^*u_1 \mapsto {}^*u_2 \quad \text{iff} \quad \begin{array}{l} {}^*u_1 = {}^*g_1 {}^*h_1^{-1},\ {}^*u_2 = {}^*g_2 {}^*h_2^{-1} \text{ and} \\ {}^\bullet g_1 = {}^\bullet g_2 \cdot \tilde{a},\ {}^\bullet h_1 = {}^\bullet h_2 \cdot \tilde{a} \end{array} \tag{9}
$$

is well-defined since there might be, say, another representation ${}^*u_1 = {}^*g_1'({}^*h_1')^{-1}$ which leads to a different assignment. Even if (9) were well-defined, there is no reason to expect that it should respect the groupoid structure. When ${}^\bullet\widetilde{\mathfrak{H}}$ is a group, one can say more:

Lemma 1. *If ${}^\bullet\widetilde{\mathfrak{H}}$ is a group then ${}^*u \circ {}^*v = {}^*w$ in $[\![\pi]\!]_1(\mathcal{F}_{\mathfrak{H},\Gamma}, x, \hat{x}, T)$ implies ${}^\bullet u \cdot {}^\bullet v = {}^\bullet w$ in ${}^\bullet\widetilde{\mathfrak{H}}$.*

Proof. This follows immediately since the groupoid structure of the fundamental germ is defined in terms of left multiplication on the unit space ${}^\bullet\mathsf{D}(\tilde{x}, \hat{x}, T)$. □

Proposition 11. *If ${}^\bullet\widetilde{\mathfrak{H}}$ is a group and T_1 and T_2 are diophantine transversals through $x_1, \hat{x}$ and $x_2, \hat{x}$, respectively, where x_1, x_2 belong to the same leaf L, then*

$$
[\![\pi]\!]_1(\mathcal{F}_{\mathfrak{H},\Gamma}, x_1, \hat{x}, T_1) \cong [\![\pi]\!]_1(\mathcal{F}_{\mathfrak{H},\Gamma}, x_2, \hat{x}, T_2).
$$

Proof. It is clear now that the bijection (9) is well-defined: in fact, since ${}^\bullet\widetilde{\mathfrak{H}}$ is a group, we have ${}^\bullet u_1 = {}^\bullet u_2$. From this it follows that $\mathsf{Dom}({}^*u_1) = \mathsf{Dom}({}^*u_2) \cdot \tilde{a}$, and that the bijection (9) defines a groupoid isomorphism. □

Let us return to the case of the linear foliation $\mathcal{F}_V$. Viewed as a double coset foliation, it is easy to see that if we choose transversals that are fibers, the fundamental germ so obtained is identical to that obtained using the suspension definition in §4.4. On the other hand, the definition of $[\![\pi]\!]_1$ available for double cosets allows us to use nonfiber transversals. As ${}^\bullet\widetilde{\mathfrak{H}} = {}^\bullet\mathbb{R}^p$ is indeed a group, it follows from Proposition 11 that these different possible choices of transversal will yield a fundamental germ that agrees up to isomorphism with that defined in §4.4.

We shall assume from this moment on that ${}^\bullet\widetilde{\mathfrak{H}}$ is a group. We will then not mention the base point x and the transversal T and write $[\![\pi]\!]_1(\mathcal{F}_{\mathfrak{H},\Gamma}, L, \hat{x})$ where L is the leaf along which diophantine approximations are taking place. If $\hat{x} \in L$ we write simply $[\![\pi]\!]_1(\mathcal{F}_{\mathfrak{H},\Gamma}, L)$.

We now give a "diophantine" description of ${}^*\mathsf{D}(\tilde{x}, \hat{x}, T)$, similar in spirit to that of ${}^*\mathbb{Z}^p_{\mathbf{R}}$ appearing in (4). Denote by $p : \widetilde{\mathfrak{H}} \to \mathfrak{H}$ the universal cover of $\mathfrak{H}$. Suppose that L is covered by a coset $\mathfrak{H}g$ and $\hat{g} \in \mathfrak{G}$ is an element covering $\hat{x}$. A subset $\mathcal{T}^{\hat{g}} \subset \mathfrak{G}$ is called a *local section* at $\hat{g}$ for the quotient map $\mathfrak{G} \to \mathfrak{H}\backslash\mathfrak{G}$ if $\mathcal{T}^{\hat{g}}$ maps homeomorphically onto an open subset containing $\mathfrak{H}\hat{g}$. We may assume without loss of generality that the transversal T through $\hat{x}$ lifts to a local section $\mathcal{T}^{\hat{g}}$ through $\hat{g}$. As our interest is in sequences which converge to $\hat{g}$ in $\mathcal{T}^{\hat{g}}$, we may assume also that $\mathcal{T}^{\hat{g}} = \hat{g} \cdot \mathcal{T}$ for some local section $\mathcal{T}$ about 1. Let ${}^*\mathcal{T}_\epsilon \subset {}^*\mathfrak{G}_\epsilon$ denote the set of infinitesimals which are represented by sequences in $\mathcal{T}$.

Now let ${}^*\tilde{h}$ be a diophantine approximation of $\hat{x}$ based at $\tilde{x}$ along T, which is characterized by the property that $\{p({}^*\tilde{h}) \cdot g\}$ lies in $\hat{g} \cdot {}^*\mathcal{T}_\epsilon \cdot {}^*\Gamma$. This gives the following diophantine description of ${}^*\mathsf{D}(\tilde{x}, \hat{x}, T)$:

$$
\begin{aligned}
{}^*\mathsf{D}(\tilde{x}, \hat{x}, T) \;=\; \Big\{\, {}^*\tilde{h} \in {}^*\widetilde{\mathfrak{H}} \;\Big|\; \exists\, {}^*\gamma \in {}^*\Gamma,\ {}^*\epsilon \in {}^*\mathcal{T}_\epsilon \text{ such that} \\
\hat{g}^{-1} \cdot p({}^*\tilde{h}) \cdot g \cdot {}^*\gamma = {}^*\epsilon \Big\}.
\end{aligned}
\tag{10}
$$

The element ${}^*\tilde{h}^\perp := {}^*\gamma$ associated to ${}^*\tilde{h}$ in (10) is called the *dual* of ${}^*\tilde{h}$. When $\hat{g} = g$, we let ${}^*\widetilde{\mathfrak{H}}_g := {}^*\mathsf{D}(\tilde{x}, T)$ denote the set of diophantine approximations and let ${}^*\widetilde{\mathfrak{H}}_g^\perp$ denote the set of duals. Thus if σ_g denotes the conjugation map $a \mapsto g^{-1}ag$,

$$
{}^*\widetilde{\mathfrak{H}}_g \;=\; \left\{ {}^*\tilde{h} \in {}^*\widetilde{\mathfrak{H}} \;\middle|\; \sigma_g(p({}^*\tilde{h})) \in {}^*\mathcal{T}_\epsilon \cdot {}^*\Gamma \right\}. \tag{11}
$$

In general, whether $g = \hat{g}$ or not, it follows that

$$
[\![\pi]\!]_1(\mathcal{F}_{\mathfrak{H},\Gamma}, L, \hat{x}) \;\subset\; \left\{ {}^*\tilde{u} \in {}^*\widetilde{\mathfrak{H}} \;\middle|\; \sigma_{\hat{g}}(p({}^*\tilde{u})) \in {}^*\mathcal{T}_\epsilon \cdot {}^*\Gamma \cdot {}^*\mathcal{T}_\epsilon \right\}.
$$

Note 6. Since $p^{-1}(e) \cong \pi_1\mathfrak{H}$, we have ${}^*\pi_1\mathfrak{H} < [\![\pi]\!]_1(\mathcal{F}_{\mathfrak{H},\Gamma}, L, \hat{x})$.

One can understand the description of ${}^*\widetilde{\mathfrak{H}}_g$ appearing in (11) as a nonlinear version of (4). In fact, if $\mathfrak{G}$ is a linear group of $p \times p$ matrices and $g \in \mathfrak{G}$, then one can think of ${}^*\mathbb{Z}^p_g$ as defined in (4) as the set of linear diophantine approximations of g (approximations of g by pairs of vectors with respect to linear algebra), whereas ${}^*\widetilde{\mathfrak{H}}_g$ can be thought of as a set of nonlinear diophantine approximations of g (approximations of g by pairs of matrices with respect to matrix algebra).

We now consider the horocyclic and geodesic flows on the unit tanget bundle of a riemannian surface, which are, as is widely appreciated, deep mathematical objects. It should come as no suprise that this deepness is reflected in their fundamental germs, which present the most complex and intractable diophantine algebra we have encountered thus far. In the remainder of this section, we will attempt to give the reader a feel for the complexity of these fundamental germs by walking through a sample calculation.

We restrict to the case $\mathfrak{G} = SL(2,\mathbb{R})$ and $\Gamma = SL(2,\mathbb{Z})$. See § 2.3 for the relevant notation. Consider first the case of the (positive) horocyclic flow $\mathsf{Hor} = \mathsf{Hor}^+_{\mathsf{SL}(2,\mathbb{Z})}$, that is, $\mathfrak{H} = H = H^+$. If D is the subgroup of matrices of the form

$$\begin{pmatrix} e^{s/2} & 0 \\ t & e^{-s/2} \end{pmatrix}$$

$s, t \in \mathbb{R}$, then D defines a local section about 1 so we take $\mathcal{T} = D$. Finally, since $H \cong (\mathbb{R}, +)$, we shall simplify notation by identifying r with the matrix A_r and write ${}^*\mathbb{R}_g = {}^*H_g$ for the set of diophantine approximations.

Let us consider the relatively simple choice

$$g = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

The right coset of g is

$$Hg = \left\{ \begin{pmatrix} r+\sqrt{2} & \sqrt{2}r+1 \\ 1 & \sqrt{2} \end{pmatrix} \,\middle|\, r \in \mathbb{R} \right\}.$$

Since Hg does not define a cycle in $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ it must be dense by a theorem of Hedlund [9], so we can expect from g a nontrivial set of diophantine approximations. The conjugate of H by g is

$$\sigma_g(H) = \left\{ \begin{pmatrix} 1+\sqrt{2}r & 2r \\ -r & 1-\sqrt{2}r \end{pmatrix} \,\middle|\, r \in \mathbb{R} \right\}.$$

In order to characterize the elements of ${}^*\mathbb{R}_g$, we shall need the following generalization of ${}^*\mathbb{Z}_r$. Let $\mathbb{O}$ be the ring of integers of a number field. For ${}^*r \in {}^*\mathbb{R}$, define

$$ {}^*\mathbb{O}_{^*r} \;=\; \{{}^*n \in {}^*\mathbb{O} \mid \exists\ {}^*n^\perp \in {}^*\mathbb{O} \text{ such that } {}^*r \cdot {}^*n - {}^*n^\perp \in {}^*\mathbb{R}_\epsilon\}. $$

Clearly ${}^*\mathbb{O}_{^*r}$ is a subgroup of ${}^*\mathbb{O}$.

Theorem 9. *Let $\mathbb{O}$ be the ring of integers in $\mathbb{Q}(\sqrt{2})$. Then ${}^*r \in {}^*\mathbb{R}_g$ if and only if there exists ${}^*\gamma = \begin{pmatrix} {}^*a & {}^*b \\ {}^*c & {}^*d \end{pmatrix} \in SL(2, {}^*\mathbb{Z})$ for which*

- $\sqrt{2}{}^*a + 2{}^*c,\ \sqrt{2}{}^*b + 2{}^*d \in {}^*\mathbb{O}_{^*r}$ *and* $(\sqrt{2}{}^*a + 2{}^*c)^\perp = 1 - {}^*a$, $(\sqrt{2}{}^*b + 2{}^*d)^\perp = -{}^*b$.
- ${}^*c, {}^*d \in {}^*\mathbb{Z}_{\sqrt{2}}$ *and* ${}^*c^\perp = 1 - {}^*a$, ${}^*d^\perp = 1 - {}^*b$.
- ${}^*b = -(\sqrt{2}{}^*b + 2{}^*d){}^*r$ *and* $({}^*a + (\sqrt{2}{}^*a + 2{}^*c){}^*r)({}^*d - ({}^*b + \sqrt{2}{}^*d){}^*r) = 1$.

Proof. From (11), ${}^*r \in \mathbb{R}_g$ if and only if there exists ${}^*\gamma \in {}^*\Gamma$ and ${}^*\epsilon, {}^*\delta \in {}^*\mathbb{R}_\epsilon$ with

$$ \begin{pmatrix} {}^*a(1 + \sqrt{2}{}^*r) + 2{}^*c{}^*r & {}^*b(1 + \sqrt{2}{}^*r) + 2{}^*d{}^*r \\ -{}^*a{}^*r + {}^*c(1 - \sqrt{2}{}^*r) & -{}^*b{}^*r + {}^*d(1 - \sqrt{2}{}^*r) \end{pmatrix} = \begin{pmatrix} 1 + {}^*\epsilon & 0 \\ {}^*\delta & (1 + {}^*\epsilon)^{-1} \end{pmatrix}. $$

The first and third items follow immediately. The second item follows upon noting that we may eliminate ${}^*r$ by multiplying the second row equations by $\sqrt{2}$ and adding them to the first row equations. $\square$

Theorem 9 illustrates why it is so difficult to say anything about the algebraic structure of ${}^*\mathbb{R}_g$ or $[\![\pi]\!]_1(\mathsf{Hor}, L)$. In order to determine whether the sum ${}^*r + {}^*s$ defines an element of ${}^*\mathbb{R}_g$, we must find a way to "compose" the corresponding duals ${}^*r^\perp, {}^*s^\perp \in \mathbb{R}_g^\perp$ to obtain one for their sum, and it is not even clear what this operation on matrices should be. One could reverse the logic and ask if the product ${}^*r^\perp \cdot {}^*s^\perp$ defines an element of $\mathbb{R}_g^\perp$, however this seems just as hopeless since the diophantine conditions spelled out in the statement of Theorem 9 are not stable with respect to matrix multiplication.

As for the geodesic flow, we leave it to the reader to formulate the appropriate analogue of Theorem 9 *e.g.* using the local section $\mathcal{T}$ for which

$$ {}^*\mathcal{T}_\epsilon \;=\; \left\{ \begin{pmatrix} 1 & {}^*\delta \\ {}^*\delta' & 1 + {}^*\delta{}^*\delta' \end{pmatrix} \;\middle|\; {}^*\delta, {}^*\delta' \in {}^*\mathbb{R}_\epsilon \right\}. $$

The result would be a set of diophantine conditions at least as daunting as that obtained for the horocyclic flow.

7 The fundamental germ of a locally free lie group action

The discussion here is very similar to that for a double coset, so we will be brief. Let $\mathfrak{V}$ be a Lie group of dimension k, M^n an n-manifold, $n > k$, $X \subset M^n$. Let $\theta : \mathfrak{V} \to \mathsf{Homeo}(X)$ be a locally-free representation whose orbits either coincide or are disjoint and let $\mathcal{L}_{\mathfrak{V}}$ be the associated lamination on X. Any diophantine transversal through $x, \hat{x}$ may be obtained as the intersection of $\mathcal{L}_{\mathfrak{V}}$ with a submanifold T of M^n of dimension $n - k$ such that $x, \hat{x} \in T$ and $T \cap (\theta(\mathfrak{V}) \cdot x)$ is discrete in $\theta(\mathfrak{V}) \cdot x$. As in the case of a double coset foliation, when ${}^{\bullet}\widetilde{\mathfrak{V}}$ is group,

(1) Groupoid multiplication in the fundamental germ corresponds to multiplication in ${}^{\bullet}\widetilde{\mathfrak{V}}$.

(2) If T_1, T_2 are transversals through $x_1, \hat{x}$ and $x_2, \hat{x}$ where x_1, x_2 belong to the same leaf L then

$$[\![\pi]\!]_1(\mathcal{L}_{\mathfrak{V}}, x_1, \hat{x}, T_1) \;\cong\; [\![\pi]\!]_1(\mathcal{L}_{\mathfrak{V}}, x_2, \hat{x}, T_2).$$

Accordingly we shorten to $[\![\pi]\!]_1(\mathcal{L}_{\mathfrak{V}}, L, \hat{x})$. We note also that when the linear foliation $\mathcal{F}_V$ is viewed as arising from a locally free Lie group action, the fundamental germ so obtained agrees with that of § 4.4.

Theorem 10. *Let $\Sigma = \Gamma\backslash\mathbb{H}^2$ be a compact hyperbolic surface, $\mathfrak{l} \subset \Sigma$ a geodesic lamination, $\hat{x} \in \mathfrak{l}$ and $l \subset \mathfrak{l}$ a leaf. Then*

$$[\![\pi]\!]_1(\mathfrak{l}, l, \hat{x}) \;=\; [\![\pi]\!]_1(\mathsf{Geod}_{\Gamma}, L, \hat{v})$$

where L is a leaf covering l and $\hat{v}$ is a tangent vector to l at $\hat{x}$.

Proof. This follows immediately from the fact that any diophantine approximation of $\hat{v}$ along L canonically defines a diophantine approximation of $\hat{x}$ along l and *vice verca*. □

8 Functoriality

We begin by recalling the notion of morphism in the category of laminations. A *lamination map* $F : \mathcal{L} \to \mathcal{L}'$ is a map satisfying the following conditions:

(1) For every leaf $L \subset \mathcal{L}$, there exists a leaf $L' \subset \mathcal{L}'$ with $F(L) \subset L'$.

(2) For all $x \in \mathcal{L}$, there exist open transversals $T \ni x$, $T' \ni F(x)$, such that $F(T) \subset T'$.

The projection $P : \mathcal{L} \to B$ of a suspension onto its base is a lamination map. On the other hand, let $\mathcal{F}$ be a foliation, M the underlying manifold. Then the canonical inclusion $\imath : \mathcal{F} \to M$ is a map which maps leaves into the unique leaf M, yet is not a lamination map since no open transversal of $\mathcal{F}$ is mapped into a point, an open transversal of M.

Let

$$F : (\mathcal{L}, x, \hat{x}) \longrightarrow (\mathcal{L}', x', \hat{x}')$$

be a lamination map. We say that F is *diophantine* if there exist diophantine transversals $T \ni x, \hat{x}$ and $T' \ni x', \hat{x}'$ such that $F(T) \subset T'$. Note that this condition is always satisfied if either $\mathcal{L}$ or $\mathcal{L}'$ are laminations defined by double cosets or locally free Lie group actions. Denote by L and L' the leaves containing x, x' and let $\widetilde{F} : \widetilde{L} \to \widetilde{L}'$ be the lift of the restriction $F|_L$. Let $\widetilde{T} \subset \widetilde{L}$, $\widetilde{T}' \subset \widetilde{L}'$ be the pre-images of $T \cap L$, $T' \cap L'$. Then for F diophantine there is a well-defined map

$$^*\mathsf{D}F : {}^*\mathsf{D}(\tilde{x}, \hat{x}, T) \longrightarrow {}^*\mathsf{D}(\tilde{x}', \hat{x}', T')$$

of diophantine approximations. If the assigment

$$^*u = {}^*g \cdot {}^*h^{-1} \ \longmapsto \ {}^*\mathsf{D}F({}^*g) \cdot ({}^*\mathsf{D}F({}^*h))^{-1}$$

leads to a well-defined map

$$[\![F]\!] : [\![\pi]\!]_1(\mathcal{L}, x, \hat{x}) \longrightarrow [\![\pi]\!]_1(\mathcal{L}', x', \hat{x}'),$$

we say that F is *germ*.

Proposition 12. *Let $\mathcal{L} = \widetilde{B} \times_\rho F$ be a suspension with $x, \hat{x}$ lying over $x_0 \in B$. Then the projection $\xi : (\mathcal{L}, x, \hat{x}) \to (B, x_0)$ is germ, and the induced map $[\![\xi]\!]$ is a groupoid monomorphism.*

Proof. It is clear from the definitions that ${}^*\mathsf{D}\xi$ is the inclusion

$$^*\mathsf{D}(\tilde{x}, \hat{x}) \ \subset \ {}^*\pi_1(B, x).$$

In particular, it follows that $[\![\xi]\!]$ is well-defined. Since the product in $[\![\pi]\!]_1(\mathcal{L}, \hat{x}, L)$ is induced by multiplication in ${}^*\pi_1(B, x)$, $[\![\xi]\!]$ is a groupoid homomorphism as well. □

Unfortunately, we cannot assert in general that the map $[\![F]\!]$ induced by a germ lamination map F defines a groupoid homomorphism. We now introduce a class of lamination maps which is sufficiently well-behaved so as to allow us to say more.

Let $\mathcal{F}$ be a foliation, M the underlying space of $\mathcal{F}$, and $\imath : \mathcal{F} \rightarrow M$ the inclusion. Although $\imath$ is not a lamination map, we may nevertheless define a map of diophantine approximations as follows. An element ${}^*g \in {}^*\mathsf{D}(\tilde{x}, \hat{x}, T)$, represented say by $\{g_\alpha\}$, may be regarded as made up from an equivalence class of sequence $\{\gamma_{g_\alpha}\}$ where the γ_{g_α} are homotopy classes of curves lying within L whose endpoints converge to $\hat{x}$. One may assume that there is an open disc $O \subset M$ about $\hat{x}$ such that the endpoints of these sequences lie entirely in O. By connecting their endpoints to $\hat{x}$ by a paths contained in O, we obtain a sequence of homotopy classes of curves $\{\eta_{g_\alpha}\} \subset \Pi_1(M, x, \hat{x})$ = the set of homotopy classes of paths from x and $\hat{x}$, hence a map

$$ {}^*\mathsf{D}\imath : {}^*\mathsf{D}(\tilde{x}, \hat{x}, T) \longrightarrow {}^*\Pi_1(M, x, \hat{x}), \qquad {}^*g \longmapsto \eta_{{}^*g} $$

which depends neither on O nor on the choice of connecting paths. More generally, given $\mathcal{L}$ a lamination and $\imath : \mathcal{L} \rightarrow X$ a map into a path-connected space, we may define a map ${}^*\mathsf{D}\imath : {}^*\mathsf{D}(\tilde{x}, \hat{x}, T) \rightarrow {}^*\Pi_1(X, \imath(x), \imath(\hat{x}))$. We say that the map $\imath$ is *germ* if ${}^*\mathsf{D}\imath$ induces a well-defined map

$$ [\![\imath]\!] : [\![\pi]\!]_1(\mathcal{L}, x, \hat{x}, T) \longrightarrow {}^*\pi_1(X, \imath(x)), $$

$$ {}^*u = {}^*g\,{}^*h^{-1} \longmapsto {}^*\mathsf{D}\imath({}^*g) \cdot ({}^*\mathsf{D}\imath({}^*h))^{-1}. $$

Definition 3. *Let $\mathcal{L}$ be a lamination arising from a group action, X a path connected space. A map $\imath : (\mathcal{L}, x, \hat{x}) \rightarrow (X, \imath(x), \imath(\hat{x}))$ is called a* **fidelity** *if it is germ and $[\![\imath]\!]$ is a groupoid monomorphism. We say that $\mathcal{L}$ is* **faithful** *if it has a fidelity.*

For example, by Proposition 12 any suspension is faithful, however if the underlying space of a suspension $\mathcal{L}$ is a manifold M, we shall see that it is much more useful to be able to assert that the inclusion $\mathcal{L} \hookrightarrow M$ is a fidelity.

For the remainder of the section, the base points x and $\hat{x}$ will be supressed in order to simplify notation.

Proposition 13. *Let $\mathcal{F}_V$ be the foliation of $\mathbb{T}^n$ induced by the p-plane $V \subset \mathbb{R}^n$. Then the inclusion $\imath : \mathcal{F}_V \rightarrow \mathbb{T}^n$ is a fidelity.*

Proof. Recall that for some $q \times p$ matrix $\mathbf{R}$, $[\![\pi]\!]_1(\mathcal{F}_V) = {}^*\mathbb{Z}^p_{\mathbf{R}}$. Then for ${}^*\mathbf{n} \in {}^*\mathbb{Z}^p_{\mathbf{R}}$, the map $[\![\imath]\!]$ is

$$ [\![\imath]\!]\big({}^*\mathbf{n}\big) \;=\; \big({}^*\mathbf{n}, {}^*\mathbf{n}^{\perp}\big) \;\in\; {}^*\mathbb{Z}^p \oplus {}^*\mathbb{Z}^q \;=\; {}^*\pi_1\mathbb{T}^n, $$

where ${}^*\mathbf{n}^{\perp}$ is the dual to ${}^*\mathbf{n}$. $[\![\imath]\!]$ is then clearly an injective homomorphism. $\square$

The problem of the existence of fidelities for laminations arising from group actions is interesting but seems difficult.

Conjecture 2. *Every lamination arising from a group action is faithful.*

Definition 4. *A germ lamination map $F : \mathcal{L} \to \mathcal{L}'$ is* **trained** *if $\mathcal{L}$ and $\mathcal{L}'$ are faithful, and there exist fidelities $\imath : \mathcal{L} \to X$, $\imath' : \mathcal{L}' \to X'$ and a map $f : X \to X'$ such that*

$$ {}^*f \circ [\![\imath]\!] \;=\; [\![\imath']\!] \circ [\![F]\!], \tag{12}$$

where ${}^*f = {}^*\pi_1(f)$ is the nonstandard version of the map on π_1 induced by f. The triple $(\imath, \imath', f)$ is called a **training** *for F.*

Theorem 11. *Let $F : \mathcal{L} \to \mathcal{L}'$ be a trained lamination map. Then the induced map $[\![F]\!]$ is a groupoid homomorphism.*

Proof. Let $(\imath, \imath', f)$ be a training for F. Then for all ${}^*u, {}^*v \in [\![\pi]\!]_1(\mathcal{L})$ such that ${}^*u \cdot {}^*v$ is defined we have

$$[\![\imath']\!] \circ [\![F]\!]\Big({}^*u \cdot {}^*v\Big) \;=\; [\![\imath']\!]\Big([\![F]\!]{}^*u \cdot [\![F]\!]{}^*v\Big).$$

Since $[\![\imath']\!]$ is injective, $[\![F]\!]\big({}^*u \cdot {}^*v\big) = [\![F]\!]{}^*u \cdot [\![F]\!]{}^*v$. □

Corollary 1. *Let $F : (\mathcal{F}, x) \to (\mathcal{F}', x')$ be a germ map of foliations. Suppose that the inclusions into the underlying manifolds $\imath : \mathcal{F} \to M$, $\imath' : \mathcal{F}' \to M'$ are fidelities. Then $[\![F]\!]$ is a groupoid homomorphism.*

Proof. Take $f : M \to M'$ to be F, viewed as a map on underlying manifolds. Then $(\imath, \imath', f)$ is a training. □

Corollary 2. *Any map $F : \mathcal{F}_V \to \mathcal{F}_{V'}$ of linear foliations of torii induces a homomorphism $[\![F]\!]$ of fundamental germs.*

9 The germ universal cover

We assume throughout this section that

(1) $\mathcal{L}$ is a weakly-minimal lamination arising from a group action.

(2) $x = \hat{x} \in L$ a fixed dense leaf.

We abreviate the associated fundamental germ to $[\![\pi]\!]_1(\mathcal{L})$. An ultrafilter $\mathfrak{U}$ is fixed throughout.

Let $p : \widetilde{L} \to L$ be the universal cover. A sequence $\{\tilde{x}_\alpha\} \subset \widetilde{L}$ is called $\mathcal{L}$-*convergent* if it projects to a sequence in L converging to some $\hat{x} \in \mathcal{L}$. Two $\mathcal{L}$-convergent sequences $\{\tilde{x}_\alpha\}$ and $\{\tilde{x}'_\alpha\} \subset \widetilde{L}$ are called $\mathcal{L}$-*asymptotic* if their projections converge to the same point $\hat{x}$ and if for every flowbox O in $\mathcal{L}$ about $\hat{x}$, there exists $X \in \mathfrak{U}$ such that $\tilde{x}_\alpha$ and $\tilde{x}'_\alpha$ lie in a common lift of a plaque of O, for all $\alpha \in X$. The asymptotic class corresponding to $\{\tilde{x}_\alpha\}$ is denoted ${}^\bullet\tilde{x}$; we refer to $\hat{x}$ as the *limit* of ${}^\bullet\tilde{x}$ and write $\lim {}^\bullet\tilde{x} = \hat{x}$. The set of ${}^\bullet\tilde{x}$ with limit $\hat{x}$ is denoted $\mathsf{Lim}_{\hat{x}}$.

Definition 5. *The* **germ universal cover** *of $\mathcal{L}$ with respect to L is*

$$[\![\widetilde{\mathcal{L}}]\!] = \left\{\text{classes } {}^\bullet\tilde{x} \text{ of } \mathcal{L}\text{-convergent sequences in } \widetilde{L}\right\}.$$

Note that for any $\hat{x} \in \mathcal{L}$, every G_L-diophantine approximation ${}^*g$ of $\hat{x}$ determines an element of $[\![\widetilde{\mathcal{L}}]\!]$, and the sets $\mathsf{Lim}_{\hat{x}}$ and ${}^*\mathsf{D}(\tilde{x}, \hat{x}, T)$ are in bijective correspondence, for any diophantine transversal T through $x, \hat{x}$.

Proposition 14. *Let $\mathcal{L}$ be compact and suppose that $L = \mathfrak{G}$ is a topological group for which ${}^*\tilde{b}, {}^*\tilde{c} \in {}^*\widetilde{\mathfrak{G}}$ are $\mathcal{L}$-asymptotic if and only if ${}^*\tilde{b} \cdot {}^*\tilde{c}^{-1} \in {}^*\widetilde{\mathfrak{G}}_\epsilon$. Then $[\![\widetilde{\mathcal{L}}]\!] = {}^\bullet\widetilde{\mathfrak{G}}$.*

Proof. Suppose that there is some ${}^\bullet\tilde{b} \in {}^\bullet\widetilde{\mathfrak{G}}$ represented by a sequence $\{\tilde{b}_\alpha\}$ which is not $\mathcal{L}$-convergent. Thus if $\{b_\alpha\}$ is the projection of this sequence to $\mathfrak{G} \subset \mathcal{L}$, then for all $\hat{x} \in \mathcal{L}$, $\hat{x}$ has a neighborhood $U_{\hat{x}} \subset \mathcal{L}$ for which there is no $X \in \mathfrak{U}$ with $\{b_\alpha\}|_X \subset U_{\hat{x}}$. The $U_{\hat{x}}$ cover $\mathcal{L}$ so that there is a subcover $U_{\hat{x}_1}, \dots, U_{\hat{x}_n}$; this implies that there exists a partition $X_1 \sqcup \dots \sqcup X_n$ of $\mathbb{N}$ with $\{b_\alpha\}|_{X_i} \subset U_{\hat{x}_i}$. Since $\mathfrak{U}$ is an ultrafilter, one of the X_i belongs to $\mathfrak{U}$, contradiction. Thus every element ${}^\bullet\tilde{b} \in {}^\bullet\widetilde{\mathfrak{G}}$ defines an element of $[\![\widetilde{\mathcal{L}}]\!]$. Since the relation of being $\mathcal{L}$-asymptotic coincides with differing by an infinitesimal, we are done. □

For example, if $\mathcal{F}_V$ is a linear p-foliation of a torus, $[\![\widetilde{\mathcal{F}_V}]\!] = {}^\bullet\mathbb{R}^p$.

Denote by

$${}^\bullet p : [\![\widetilde{\mathcal{L}}]\!] \longrightarrow \mathcal{L}$$

the natural projection defined ${}^\bullet\tilde{x} \mapsto \lim {}^\bullet\tilde{x}$. The *leaf* $L_{{}^\bullet\tilde{x}}$ through ${}^\bullet\tilde{x}$ is defined to be the set of ${}^\bullet\tilde{y}$ such that the following is true: there are representative sequences $\{\tilde{x}_\alpha\}$, $\{\tilde{y}_\alpha\}$, and paths $\tilde{\eta}_\alpha$ connecting $\tilde{x}_\alpha$ to $\tilde{y}_\alpha$ so that $p(\tilde{\eta}_\alpha)$ converges to a path connecting $\hat{x} = \lim {}^\bullet\tilde{x}$ to $\hat{y} = \lim {}^\bullet\tilde{y}$. Note that in particular we have $L_{\hat{y}} = L_{\hat{x}}$.

Theorem 12. *$[\![\widetilde{\mathcal{L}}]\!]$ may be given the structure of a lamination whose leaves are nowhere dense and for which ${}^\bullet p$ is an open lamination map.*

Proof. Denote by $[\![T]\!] \subset [\![\widetilde{\mathcal{L}}]\!]$ the pre-image of a transversal $T \subset \mathcal{L}$ and well-order each $\mathsf{Lim}_{\hat{x}}$ for $\hat{x} \in T$. Note that the cardinalities of the $\mathsf{Lim}_{\hat{x}}$ are the same: that of the continuum, since L is dense and $L \cap T$ is countable. We define a decomposition

$$[\![T]\!] = \bigsqcup T_\alpha \tag{13}$$

where T_α is the section over T defined by $\hat{x} \mapsto$ the αth element of $\mathsf{Lim}_{\hat{x}}$. By definition of the leaves of $[\![\widetilde{\mathcal{L}}]\!]$, given $\hat{x}, \hat{y} \in T$,

$$\left(\bigcup_{{}^\bullet\tilde{x} \in \mathsf{Lim}_{\hat{x}}} L_{{}^\bullet\tilde{x}} \right) \bigcap \left(\bigcup_{{}^\bullet\tilde{y} \in \mathsf{Lim}_{\hat{y}}} L_{{}^\bullet\tilde{y}} \right) \neq \emptyset \tag{14}$$

if and only if $L_{\hat{x}} = L_{\hat{y}}$, and in the latter event the two unions of leaves appearing in (14) are equal. Since for any $\hat{x}$, $T \cap L_{\hat{x}}$ is countable, we may choose the ordering of each $\mathsf{Lim}_{\hat{y}}$, $\hat{y} \in T \cap L_{\hat{x}}$, so that all of the αth elements lie on distinct leaves. In this way we may asume that the associated section T_α intersects any leaf of $[\![\widetilde{\mathcal{L}}]\!]$ no more than once. We topologize each section T_α through its identification with T, and give $[\![\widetilde{\mathcal{L}}]\!]$ the associated product lamination structure. By construction of this topology, ${}^\bullet p$ becomes an open lamination map. $\square$

The topology constructed in Theorem 12 is called a *germ universal cover topology*: it is not unique and depends on the choice of decomposition (13). From now on, we assume that $[\![\widetilde{\mathcal{L}}]\!]$ has been equipped with such a topology.

There is a canonical simply connected leaf corresponding to the inclusion $\widetilde{L} \hookrightarrow [\![\widetilde{\mathcal{L}}]\!]$. In particular,

$$[\![\pi]\!]_1([\![\widetilde{\mathcal{L}}]\!], x) = 0$$

for any $x \in \widetilde{L}$. Thus $[\![\widetilde{\mathcal{L}}]\!]$ can be thought of as the ordinary universal cover $\widetilde{L}$ surrounded by a nonstandard cloud of leaves corresponding to the laminar accumulations of L; since these leaves are nowhere dense, one might say that on passing to $[\![\widetilde{\mathcal{L}}]\!]$ all of the diophantine approximations within $\mathcal{L}$ have been "unwrapped".

We now posit $[\![\widetilde{\mathcal{L}}]\!]$ as the unit space of an enhanced groupoid structure for $[\![\pi]\!]_1(\mathcal{L})$. Let ${}^*u \in [\![\pi]\!]_1(\mathcal{L})$ and ${}^\bullet\tilde{x} \in [\![\widetilde{\mathcal{L}}]\!]$. We say that ${}^*u$ *acts* on ${}^\bullet\tilde{x}$ if there exist representative sequences such that $\{u_\alpha \cdot \tilde{x}_\alpha\}$ defines an $\mathcal{L}$-convergent sequence ${}^*u \cdot {}^\bullet\tilde{x}$ with

$$\lim({}^*u \cdot {}^\bullet\tilde{x}) = \lim {}^\bullet\tilde{x}.$$

Defining the domain $\mathsf{Dom}({}^*u)$ and range $\mathsf{Ran}({}^*u)$ of ${}^*u$ through this notion of action, we see that $[\![\widetilde{\mathcal{L}}]\!]$ yields a new groupoid structure on $[\![\pi]\!]_1(\mathcal{L})$, called the *geometric groupoid structure*. It is clear that both $\mathsf{Dom}(^*u)$ and $\mathsf{Ran}(^*u)$ are sublaminations of $[\![\, \widetilde{\mathcal{L}} \,]\!]$, since ${}^\bullet\tilde{x} \in \mathsf{Dom}(^*u)$ implies that $L_{^\bullet\tilde{x}} \subset \mathsf{Dom}(^*u)$. Thus we may view $[\![\pi]\!]_1(\mathcal{L})$ as a groupoid of bijections between sublaminations of $[\![\, \widetilde{\mathcal{L}} \,]\!]$. Note that the unit space for the old groupoid structure, ${}^\bullet\mathsf{D}(\tilde{x}, T)$, maps into the new unit space $[\![\, \widetilde{\mathcal{L}} \,]\!]$ via its bijection with Lim_x. There is a canonical inclusion of the old groupoid structure into the geometric groupoid structure, given by extension of domain and range, however in general this map need not be a groupoid homomorphism.

Assumption. For the remainder of the paper, we will assume that $[\![\pi]\!]_1(\mathcal{L})$ is endowed with the geometric groupoid structure.

Definition 6. *We say that $[\![\pi]\!]_1(\mathcal{L})$ is* **tame** *if whenever $\lim {}^\bullet\tilde{x} = \lim {}^\bullet\tilde{y}$, there exists $^*u \in [\![\pi]\!]_1(\mathcal{L})$ such that $^*u \cdot {}^\bullet\tilde{x} = {}^\bullet\tilde{y}$.*

Proposition 15. *If $[\![\pi]\!]_1(\mathcal{L})$ is tame, then the quotient*

$$[\![\pi]\!]_1(\mathcal{L}) \Big\backslash [\![\, \widetilde{\mathcal{L}} \,]\!]$$

is homeomorphic to $\mathcal{L}$.

Proof. The equivalence relation enacted by the action of $[\![\pi]\!]_1(\mathcal{L})$ identifies precisely those points of $[\![\, \widetilde{\mathcal{L}} \,]\!]$ which map to the same point $\hat{x} \in \mathcal{L}$ by ${}^\bullet p$. Since ${}^\bullet p$ is open, it follows that quotient topology is that the of $\mathcal{L}$. □

Theorem 13. *If $[\![\pi]\!]_1(\mathcal{L})$ is tame and a group, then there is a germ universal cover topology on $[\![\, \widetilde{\mathcal{L}} \,]\!]$ for which $[\![\pi]\!]_1(\mathcal{L})$ acts as a group of homeomorphisms.*

Proof. Let $[\![T]\!]$ be the preimage of a transversal T of $\mathcal{L}$. As $[\![\pi]\!]_1(\mathcal{L})$ is a group, $\mathsf{Dom}(^*u) = [\![\, \widetilde{\mathcal{L}} \,]\!]$ for every element $^*u \in [\![\pi]\!]_1(\mathcal{L})$, and moreover $^*u([\![T]\!]) = [\![T]\!]$. Let $i : T \to [\![T]\!]$ be a section so that for all ${}^\bullet\tilde{x} \in [\![\, \widetilde{\mathcal{L}} \,]\!]$, $i(T) \cap L_{^\bullet\tilde{x}}$ contains at most one point. Since $[\![\pi]\!]_1(\mathcal{L})$ acts without fixed points and is tame, we have a decomposition as disjoint union

$$[\![T]\!] = \bigsqcup_{^*u \in [\![\pi]\!]_1(\mathcal{L})} {}^*u(i(T)).$$

Now construct as in Theorem 12 a lamination structure on $[\![\, \widetilde{\mathcal{L}} \,]\!]$ based on this decomposition. It follows then that each $^*u \in [\![\pi]\!]_1(\mathcal{L})$ acts homeomorphically on $[\![\, \widetilde{\mathcal{L}} \,]\!]$. □

Proposition 16. *Let $F : (\mathcal{L}, L) \to (\mathcal{L}', L')$ be a lamination map, where L and L' are dense leaves. Then F induces a map*

$$[\![\, \widetilde{F} \,]\!] : [\![\, \widetilde{\mathcal{L}} \,]\!] \longrightarrow [\![\, \widetilde{\mathcal{L}'} \,]\!],$$

continuous with respect to appropriate choices of germ universal cover topologies.

Proof. Denote by $p' : \widetilde{L}' \to L'$ the universal cover. The map $[\![\widetilde{F}]\!]$ is defined by representing ${}^{\bullet}\tilde{x}$ by a sequence $\{\tilde{x}_\alpha\}$ and taking $[\![\widetilde{F}]\!]({}^{\bullet}\tilde{x})$ to be the asymptotic class of $\{\widetilde{F}(\tilde{x}_\alpha)\}$. Now let $[\![\tau']\!]$ be any germ universal cover topology on $[\![\widetilde{\mathcal{L}}']\!]$, say constructed from a transversal T'. Since F is a lamination map, there exists a transversal T with $F(T) \subset T'$. We may thus find a decomposition $[\![T]\!] = \sqcup T_\alpha$ compatible with that of $[\![T']\!]$ *i.e.* so that $[\![\widetilde{F}]\!](T_\alpha) \subset T'_\alpha$ for all α. Let $[\![\tau]\!]$ to be the associated germ universal cover topology. Then $[\![\widetilde{F}]\!]$ is continuous with respect to $[\![\tau]\!]$ and $[\![\tau']\!]$. □

We now return to the question of functoriality, which we must address in view of our adoption of a new groupoid structure. If we reconsider the notions of fidelities and trainings with regard to the geometric groupoid structure, then the analogue of Theorem 11 – as well as its corollaries – remain true with identical proofs. For the remainder of the paper, the concepts of fidelity and training will be understood in the context of the geometric groupoid structure.

The classical universal cover enjoys the property that the lift $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ of a map $f : X \to Y$ is $\pi_1 X$-equivariant. We now describe conditions under which the same can be said for a lamination map. A germ lamination map $F : \mathcal{L} \to \mathcal{L}'$ is said to be *geometric* if for all ${}^{*}u \in [\![\pi]\!]_1(\mathcal{L})$, $[\![\widetilde{F}]\!](\mathsf{Dom}({}^{*}u)) \subset \mathsf{Dom}\big([\![F]\!]({}^{*}u)\big)$, $[\![F]\!] : [\![\pi]\!]_1(\mathcal{L}) \to [\![\pi]\!]_1(\mathcal{L}')$ is a homomorphism and

$$[\![\widetilde{F}]\!]({}^{*}u \cdot {}^{\bullet}\tilde{x}) \;=\; [\![F]\!]({}^{*}u) \cdot [\![\widetilde{F}]\!]({}^{\bullet}\tilde{x}).$$

Examples of geometric maps are the projection $\mathcal{L}_\rho \to B$ of a suspension onto its base and any map of manifolds $f : M \to M'$.

We say that a lamination $\mathcal{L}$ is *geometrically faithful* if it has a *geometric fidelity*: a fidelity $\imath : \mathcal{L} \to X$ which is geometric and for which $[\![\tilde{\imath}]\!] : [\![\widetilde{\mathcal{L}}]\!] \to [\![\widetilde{X}]\!]$ is injective. In addition $F : \mathcal{L} \to \mathcal{L}'$ is said to be *geometrically trained* if it possesses a training $(\imath, \imath', f)$ where $\imath, \imath'$ are geometric fidelities. For example, the fidelity $\imath : \mathcal{F}_V \to \mathbb{T}^n$ of a linear foliation of a torus is geometric, as well as the projection of a suspension onto a compact base.

Theorem 14. *Let $F : \mathcal{L} \to \mathcal{L}'$ be geometrically trained. Then F is geometric.*

Proof. Let $(\imath, \imath', f)$ be a geometric training. Then we have

$$[\![\tilde{\imath}]\!] \circ [\![\widetilde{F}]\!]({}^{*}u \cdot {}^{\bullet}\tilde{x}) \;=\; [\![\tilde{\imath}]\!]\big([\![F]\!]({}^{*}u) \cdot [\![\widetilde{F}]\!]({}^{\bullet}\tilde{x})\big)$$

which implies the result as $[\![\tilde{\imath}]\!]$ is injective. □

Corollary 3. *Suppose $F : \mathcal{F} \to \mathcal{F}'$ is a lamination map of foliations such that the inclusions into the underlying manifolds are geometric fidelities. Then F is geometric. In particular any lamination map of linear foliations of torii is geometric.*

10 Covering space theory

A surjective lamination map $P : \mathcal{L} \to \mathcal{L}'$ is called a *lamination covering* if $P|_L$ is a covering map for every leaf $L \subset \mathcal{L}$. A lamination map which is a covering map in the classical sense is a lamination covering but not all lamination coverings occur this way *e.g.* the projection $\xi : \mathcal{L} \to B$ of a suspension onto its base. We say that P is *cover trained* if it has a training (ι, ι', p) in which $p : X \to X'$ is a covering map.

Theorem 15. *Let $P : \mathcal{L} \to \mathcal{L}'$ be a germ lamination covering that is cover trained. Then*

(1) *The induced map of fundamental germs*

$$[\![P]\!] : [\![\pi]\!]_1(\mathcal{L}) \longrightarrow [\![\pi]\!]_1(\mathcal{L}')$$

is a groupoid monomorphism.

(2) *The induced map of germ universal covers*

$$[\![\, \widetilde{P} \,]\!] : [\![\, \widetilde{\mathcal{L}} \,]\!] \longrightarrow [\![\, \widetilde{\mathcal{L}}' \,]\!]$$

is an open, injective map with respect to appropriate choices of germ universal cover topologies.

Proof. The first statement follows from the definition of training and the fact that ${}^*p$ is injective on ${}^*\pi_1$. Let L, L' be dense leaves in $\mathcal{L}, \mathcal{L}'$ containing x, x'. Then the lift of the restriction $P|_L$, $\widetilde{P}|_L : \widetilde{L} \to \widetilde{L}'$, is a homeomorphism. It follows that the induced map $[\![\, \widetilde{P} \,]\!]$ is injective. $[\![\, \widetilde{P} \,]\!]$ is automatically open with respect to the germ universal cover topologies constructed as in Proposition 16. □

Note 7. Here is an example when the map $[\![\, \widetilde{P} \,]\!]$ is not surjective. Take $\mathcal{L} = \mathbb{R}$, $\mathcal{L}' = \mathbb{S}^1$ and $P : \mathbb{R} \to \mathbb{S}^1$ the universal cover. Then $[\![\, \widetilde{\mathcal{L}} \,]\!] \approx \mathbb{R}$ but $[\![\, \widetilde{\mathcal{L}}' \,]\!] \approx {}^\bullet\mathbb{R}$.

Thus when P is a cover trained, the image

$$\mathsf{C} = [\![P]\!]\Big([\![\pi]\!]_1(\mathcal{L})\Big)$$

is a subgroupoid of $[\![\pi]\!]_1(\mathcal{L}')$. We shall now construct lamination coverings from subgroups, restricting attention to the case where $[\![\pi]\!]_1(\mathcal{L})$ is tame and a group. Assume that $[\![\widetilde{\mathcal{L}}]\!]$ has been given a germ universal cover topology $[\![\tau]\!]$ of the type guaranteed by Theorem 13. Consider a subgroup $\mathsf{C} < [\![\pi]\!]_1(\mathcal{L})$ and denote by $\mathcal{L}_{\mathsf{C}}$ the quotient $\mathsf{C}\backslash[\![\widetilde{\mathcal{L}}]\!]$.

Theorem 16. *$\mathcal{L}_{\mathsf{C}}$ is a lamination and the map $\mathcal{L}_{\mathsf{C}} \to \mathcal{L}$ is a lamination covering.*

Proof. The first statement follows immediately from the fact that C is a subgroup of $[\![\pi]\!]_1(\mathcal{L})$. By construction, $\mathcal{L}_{\mathsf{C}} \to \mathcal{L}$ is surjective and a covering when restricted to any leaf. $\square$

Two lamination coverings $P_i : \mathcal{L}_i \to \mathcal{L}$, $i = 1, 2$, are *isomorphic* if there exists a geometric homeomorphism $F : \mathcal{L}_1 \to \mathcal{L}_2$ such that $P_1 = P_2 \circ F$. The group of automorphisms of a lamination cover P is denoted $\mathsf{Aut}(P)$.

Proposition 17. *Let $^*u \in [\![\pi]\!]_1(\mathcal{L}, x)$ and $\mathsf{C}' = {^*u} \cdot \mathsf{C} \cdot {^*u}^{-1}$. Then $\mathcal{L}_{\mathsf{C}} \to \mathcal{L}$ and $\mathcal{L}_{\mathsf{C}'} \to \mathcal{L}$ are isomorphic.*

Proof. The bijection $^\bullet\tilde{x} \mapsto {^*u} \cdot {^\bullet\tilde{x}}$ defines a homeomorphism $[\![\widetilde{\mathcal{L}}]\!] \to [\![\widetilde{\mathcal{L}}]\!]$ which descends to an isomorphism of covers. $\square$

Now suppose $\mathsf{C} \lhd [\![\pi]\!]_1(\mathcal{L}, x)$ is a normal subgroup and $P_{\mathsf{C}} : \mathcal{L}_{\mathsf{C}} \to \mathcal{L}$ the associated covering.

Theorem 17. *$\mathsf{Aut}(P_{\mathsf{C}})$ is isomorphic to the quotient $[\![\pi]\!]_1(\mathcal{L}, x)/\mathsf{C}$. The quotient of $\mathcal{L}_{\mathsf{C}}$ by $[\![\pi]\!]_1(\mathcal{L}, x)/\mathsf{C}$ is $\mathcal{L}$.*

Proof. Every element of $\mathcal{L}_{\mathsf{C}}$ is a class $\mathsf{C} \cdot {^\bullet\tilde{x}}$, for $^\bullet\tilde{x} \in [\![\widetilde{\mathcal{L}}]\!]$. The action of $[\![\pi]\!]_1(\mathcal{L}, x) / \mathsf{C}$ on such classes is well-defined and yields a subgroup of $\mathsf{Aut}(P_{\mathsf{C}})$. On the other hand, the set $\mathsf{Lim}_{\hat{x}}$ is a $[\![\pi]\!]_1(\mathcal{L})$-set on which any geometric automorphism acts automorphically. However the automorphism group of $\mathsf{Lim}_{\hat{x}}$ is $[\![\pi]\!]_1(\mathcal{L}, x)/\mathsf{C}$, so it follows that $\mathsf{Aut}(P_{\mathsf{C}}) \subset [\![\pi]\!]_1(\mathcal{L}, x)/\mathsf{C}$. It is clear that the quotient of $\mathcal{L}_{\mathsf{C}}$ by $[\![\pi]\!]_1(\mathcal{L}, x)/\mathsf{C}$ is $\mathcal{L}$. $\square$

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